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Comisión de Pedagogía – Topología algebraica
Daniela Egas Santander, Martín Gonzalez,
Marc Stephan, Sinan Yalin

INTRODUCTION TO ALGEBRAIC TOPOLOGY

Contents

1. Computing tools in singular homology

Computing singular homology with the tools presented until now can prove to be quite difficult. In this section, we will discover how to compute homology groups in a pleasant and very intuitive way using powerful tools like the Mayer-Vietoris long exact sequence and the Excision theorem. We will not prove these statements (the proofs use a very deep - but technical - theorem known as the Smalls Chains Theorem) and rather show how to use them on basic examples. Then, we will enounce the Smalls Chains Theorem and give some further applications which will be useful in the last part of the course, where we will state and prove the Classification Theorem.

1.1. **The Mayer-Vietoris long exact sequence.**

Theorem 1.1. *Let X be a topological space and let U, V two open subsets covering X. Then there is a functorial long exact sequence of abelian groups*

$$\cdots \longrightarrow H_i(U\cap V) \longrightarrow H_i(U)\oplus H_i(V) \longrightarrow H_i(X) \longrightarrow H_{i-1}(U\cap V)$$

$$\longrightarrow H_{i-1}(U)\oplus H_{i-1}(V) \longrightarrow H_{i-1}(X) \longrightarrow \cdots \longrightarrow H_0(X) \longrightarrow 0$$

called the Mayer-Vietoris sequence associated to X with respect to the pair of open subsets (U, V).

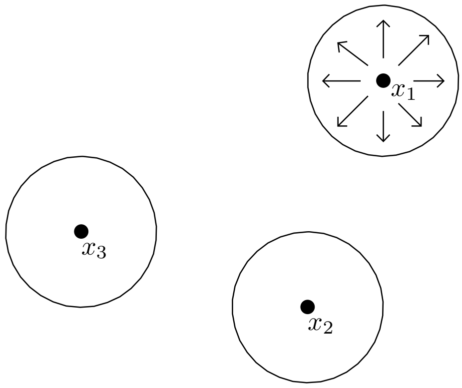
We now explain how to use the Mayer-Vietoris sequence with a first example.

Example 1.2 (**Homology groups $\mathbb{R}^n$ and $\mathbb{S}^n$ minus points and lines**)**.** Let us compute some basic homology groups using the Mayer-Vietoris long exact sequence:

- **Homology groups of $\mathbb{R}^n - \{p_0$ points$\}$:** Let $x_1, \ldots, x_{p_0}$ be p_0 points in $\mathbb{R}^n$. Let us set

$$\mathbb{R}^n = U\cup V := \left(\mathbb{R}^n \backslash \coprod_{i=1}^{p_0} x_i\right) \bigcup \left(\coprod_{i=1}^{p_0} B_i\right)$$

where the B_i are pairwise disjoint open balls with centers each x_i.



The Mayer-Vietoris sequence reads

$$\begin{gathered}0 \longrightarrow H_{n+1}(\mathbb{R}^n) \longrightarrow H_n(U \cap V) \longrightarrow H_n(U) \oplus H_n(V) \longrightarrow H_n(\mathbb{R}^n) \longrightarrow \\ \longrightarrow H_{n-1}(U \cap V) \longrightarrow H_{n-1}(U) \oplus H_{n-1}(V) \longrightarrow H_{n-1}(\mathbb{R}^n) \longrightarrow \cdots \\ \cdots H_1(U \cap V) \longrightarrow H_1(U) \oplus H_1(V) \longrightarrow H_1(\mathbb{R}^n) \longrightarrow \\ H_0(U \cap V) \longrightarrow H_0(U) \oplus H_0(V) \longrightarrow H_0(\mathbb{R}^n) \longrightarrow 0.\end{gathered}$$

In this case scenario, we are searching for $H_*(U)$ which we find by the following calculus:

– First of all, as $\mathbb{R}^n$ is contractible, we have

$$H_*(\mathbb{R}^n) \simeq \left\{ \begin{array}{ll} 0 & \text{if } i > 0 \\ \mathbb{Z} & i = 0 \end{array} \right. .$$

– Next, as the B_i are pairwise disjoint, we have

$$H_*(V) = H_*\left(\coprod_{i=1}^{p_0} B_i\right) \simeq \bigoplus_{i=1}^{p_0} H_*(B_i).$$

As each B_i is a contractible ball, for all $1 \leqslant i \leqslant p_0$, we have

$$H_*(B_i) \simeq \left\{ \begin{array}{ll} 0 & \text{if } i > 0 \\ \mathbb{Z} & i = 0 \end{array} \right. .$$

Thus, we obtain

$$H_*(V) \simeq \left\{ \begin{array}{ll} 0 & \text{if } i > 0 \\ \mathbb{Z}^{p_0} & i = 0 \end{array} \right. .$$

– Finally, $U \cap V$ is homotopically equivalent to a disjoint sum of spheres

$$U \cap V \sim_h \coprod_{i=1}^{p_0} \mathbb{S}^{n-1},$$

so its homology groups are

$$H_*(U \cap V) \simeq \left\{ \begin{array}{ll} \mathbb{Z}^{p_0} & i = 0, n-1 \\ 0 & \text{else} \end{array} \right. .$$

Then the Mayer-Vietoris sequence is now reduced to

$$\begin{gathered}0 \longrightarrow 0 \longrightarrow H_n(U) \oplus 0 \longrightarrow 0 \longrightarrow \\ \longrightarrow \mathbb{Z}^{p_0} \longrightarrow H_{n-1}(U) \oplus 0 \longrightarrow 0 \longrightarrow \cdots \\ \cdots 0 \longrightarrow H_1(U) \oplus 0 \longrightarrow 0 \longrightarrow \\ \mathbb{Z}^{p_0} \longrightarrow H_1(U) \oplus \mathbb{Z}^{p_0} \longrightarrow \mathbb{Z} \longrightarrow 0.\end{gathered}$$

By exactness of the sequence, we have

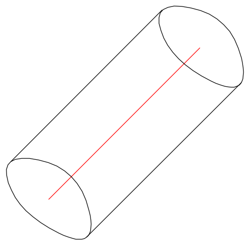
– $H_k(U) = 0$ if $k \geqslant n$;
– $H_{n-1}(U) \simeq H_{n-1}(U \cap V) \simeq \mathbb{Z}^{p_0}$;

– As $\mathbb{Z}$ is abelian free, the s.e.s. $0 \longrightarrow \mathbb{Z}^{p_0} \longrightarrow H_1(U) \oplus \mathbb{Z}^{p_0} \longrightarrow \mathbb{Z} \longrightarrow 0$ has a section so we have $H_0(U) \oplus \mathbb{Z}^{p_0} \simeq \mathbb{Z} \oplus \mathbb{Z}^{p_0}$. We conclude that $H_0(U) \simeq \mathbb{Z}$.

· **Homology groups of** $\mathbb{R}^3 - \{p_1$**disjoint lines**$\}$**:** Let $L_1, \dots, L_{p_1}$ be p_1 disjoint lines in $\mathbb{R}^3$. Let us set

$$\mathbb{R}^3 = U \cup V := \left(\mathbb{R}^3 \backslash \coprod_{i=1}^{p_1} L_i\right) \bigcup \left(\coprod_{i=1}^{p_1} C_i\right),$$

where the C_i are pairwise disjoint (filled) cylinders with souls each line L_i, as illustrated below



The Mayer-Vietoris sequence reads

$$\begin{aligned} 0 \longrightarrow H_4(\mathbb{R}^3) \longrightarrow H_3(U \cap V) \longrightarrow H_3(U) \oplus H_3(V) \longrightarrow H_3(\mathbb{R}^3) \longrightarrow \\ \longrightarrow H_2(U \cap V) \longrightarrow H_2(U) \oplus H_2(V) \longrightarrow H_2(\mathbb{R}^3) \longrightarrow \\ \longrightarrow H_1(U \cap V) \longrightarrow H_1(U) \oplus H_1(V) \longrightarrow H_1(\mathbb{R}^3) \longrightarrow \\ H_0(U \cap V) \longrightarrow H_0(U) \oplus H_0(V) \longrightarrow H_0(\mathbb{R}^3) \longrightarrow 0. \end{aligned}$$

Again, we are searching for $H_*(U)$, which we find by the following calculus:

– As the C_i are pairwise disjoint, we have

$$H_*(V) = H_*\left(\coprod_{i=1}^{p_0} C_i\right) \simeq \bigoplus_{i=1}^{p_0} H_*(C_i).$$

As each B_i is a contractible cylinder, for all $1 \leqslant i \leqslant p_1$, we have

$$H_*(V) \simeq \left\{ \begin{array}{ll} 0 & \text{if } i > 0 \\ \mathbb{Z}^{p_1} & i = 0 \end{array} \right. .$$

– Finally, $U \cap V$ is homotopically equivalent to a disjoint sum of spheres $\mathbb{S}^1$ so its homology groups are

$$H_*(U \cap V) \simeq \left\{ \begin{array}{ll} \mathbb{Z}^{p_1} & i = 1 \\ \mathbb{Z}^{p_1} & i = 0 \\ 0 & \text{else} \end{array} \right. .$$

Then the Mayer-Vietoris sequence give us

$$H_*(U) \simeq \left\{ \begin{array}{ll} \mathbb{Z}^{p_1} & i = 1 \\ \mathbb{Z} & i = 0 \\ 0 & \text{else} \end{array} \right. .$$

· **Homology groups of** $\mathbb{R}^3 - \{p_0$ **points and** p_1**disjoint lines**$\}$**:** Let $A_1, \dots, A_{p_0}$ be p_0 different points in $\mathbb{R}^3$ and $L_1, \dots, L_{p_1}$ be p_1 disjoint lines in $\mathbb{R}^3$ which do not contain any of the p_0 points. Let us set

$$\mathbb{R}^3 = U \cup V := \left(\mathbb{R}^3 \backslash \coprod_{i=1}^{p_0} A_i\right) \bigcup \left(\mathbb{R}^3 \backslash \coprod_{i=1}^{p_1} L_i\right)$$

and set B_i and C_i as before. In this case scenario, we are searching for $H_*(U \cap V)$. It is then not difficult, in light of the calculations done earlier, to find

$$H_*(U \cap V) \simeq \left\{ \begin{array}{ll} \mathbb{Z} & i = 0 \\ \mathbb{Z}^{p_1} & i = 1 \\ \mathbb{Z}^{p_0} & i = 2 \\ 0 & \text{else} \end{array} \right. .$$

- **The case of the sphere:** Let p be a point in $\mathbb{S}^n$. We have $\mathbb{S}^n - \{p\} \sim \mathbb{R}^n$. Thus,
$$H_*(\mathbb{S}^n - \{p_0 \,\text{points}\}) \simeq H_*(\mathbb{R}^n - \{(p_0-1)\,\text{points}\}).$$
and we get
$$H_*(\mathbb{S}^3 - \{p_0 \,\text{points and}\, p_1 \,\text{lines}\}) \simeq \begin{cases} \mathbb{Z}^{(p_0-1)} & \text{if } i=2 \\ \mathbb{Z}^{p_1} & \text{if } i=1 \\ \mathbb{Z} & \text{if } i=0 \\ 0 & \text{else} \end{cases}$$

One further and important example is the following.

1.1.1. *Homology groups of the projective plane.* Let us consider the projective plane $\mathbb{R}P^2 := \mathbb{S}^2/\{\pm 1\}$ which is known (e.g. the first part of the course) to be isomorphic to
$$M \bigcup_{\partial\mathbb{D}^2=\mathbb{S}^1=\partial M} \mathbb{D}^2$$
, where M is the Moebius band. Notice that the border ∂M of M is a circle.

Proposition 1.3. *The projective plane $\mathbb{R}P^2$ has homology groups:*
$$H_i(\mathbb{R}P^2) \simeq \begin{cases} \mathbb{Z} & \text{if } i=0 \\ \mathbb{Z}/2\mathbb{Z} & \text{if } i=1 \\ 0 & \text{if } i=2 \\ 0 & \text{else} \end{cases}$$

Proof. We are going to apply the Mayer-Vietoris sequence to the open subspaces M and $\mathbb{D}^2$. First of all, $H_i(M \cap \mathbb{D}^2)$ is evidently isomorphic to $H_i(\mathbb{S}^1)$. The group $H_i(M)$ is also isomorphic to $H_i(\mathbb{S}^1)$ as M can be deformation retracted to its soul, which is nothing more than $\mathbb{S}^1$. Now, the map $d_1 : H_1(M \cap \mathbb{D}^2) \longrightarrow H_1(M) \oplus H_1(\mathbb{D}^2) \simeq H_1(M)$ is the multiplication by two, as the border of the Moebius band, when retracted to the soul of M, makes the complete path along the circle twice. The Mayer-Vietoris sequence is then
$$0 \longrightarrow H_2(\mathbb{R}P^2) \longrightarrow \mathbb{Z} \xrightarrow{\times 2} \mathbb{Z} \longrightarrow H_1(\mathbb{R}P^2) \longrightarrow \mathbb{Z} \xrightarrow{d_2} \mathbb{Z}\oplus\mathbb{Z} \xrightarrow{d_3} H_0(\mathbb{R}P^2) \longrightarrow 0.$$
Now, as the map d_3 is surjective and the map d_2 is injective, we have $H_0(\mathbb{R}P^2) \simeq \mathbb{Z}$. Next, as d_1 is the multiplication by 2, it is injective so $H_2(\mathbb{R}P^2)$, being isomorphic to the kernel of d_1, is zero. Finally, as $H_1(\mathbb{R}P^2)$ is isomorphic to the quotient of $\mathbb{Z}$ by the image of d_1, which is the subgroup $2\mathbb{Z}$, we get $H_1(\mathbb{R}P^2) \simeq \mathbb{Z}/2\mathbb{Z}$. □

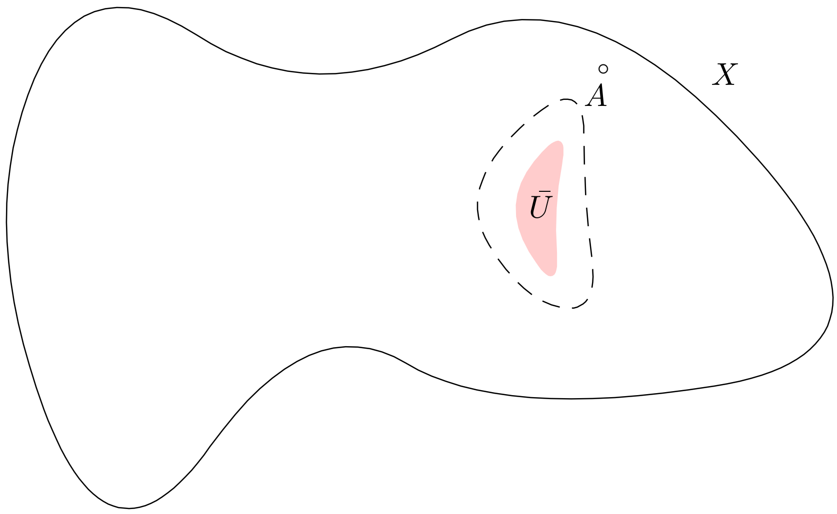
Remark 1.4. One can ask: what happens if we look for homology with coefficients other than in $\mathbb{Z}$? We can indeed generalize chain complexes of a simplicial complex K for an arbitrary abelian group G by setting
$$C_i(X, G) := G[K^{(i)}] = G \otimes_{\mathbb{Z}} C_i(K)$$
Then homology with coefficients can be defined by $H_i(K, G) := Z_i(K, G)/B_i(K, G)$ for all $i \geqslant 0$. All the results we used in this course (Excision, Mayer-Vietoris etc..) can be generalized to this case (but we have to be careful : in general, we have $H_i(X, G) \neq G \otimes_{\mathbb{Z}} H_i(K)$ for $i \neq 1$). See Hatcher's book *Algebraic Topology* for a proper introduction about this topic.

Going back to the case of $\mathbb{R}P^2$, if we take $G = \mathbb{Z}$, we get the example above. If we take $G = \mathbb{Z}/2\mathbb{Z}$, then d_1, which is the multiplication by two, is now the zero map. This means that
$$H_0(\mathbb{R}P^2, \mathbb{Z}/2\mathbb{Z}) \simeq \mathbb{Z}/2\mathbb{Z}, H_1(\mathbb{R}P^2, \mathbb{Z}/2\mathbb{Z}) \simeq \mathbb{Z}/2\mathbb{Z}, \text{ and } H_2(\mathbb{R}P^2, \mathbb{Z}/2\mathbb{Z}) \simeq \mathbb{Z}/2\mathbb{Z}.$$
If we take $G = \mathbb{Q}$, then d_1, which is the multiplication by two, is now an isomorphism (with as inverse the map $a \mapsto \frac{a}{2}$). This means that
$$H_0(\mathbb{R}P^2, \mathbb{Q}) \simeq \mathbb{Q}, \text{ and } H_1(\mathbb{R}P^2, \mathbb{Q}) \simeq H_2(\mathbb{R}P^2, \mathbb{Q}) \simeq 0.$$

We can notice that changing coefficients also changes the resulting homology groups. In particular, there is an interesting theory of Betti numbers and Euler characteristic *with coefficients* which we will not detail in this course.
Let us now introduce the second main tool we will use to compute homology gorups.

1.2. **The Excision theorem.**

Theorem 1.5. *Let X be a topological space, A a subspace of X and $U \subset A$ a subspace of A such that $\overline{U} \subset \mathring{A}$.*



Then, the inclusion of topological pairs $(X - U, A - U) \longrightarrow (X, A)$ induces an isomorphism in relative homology, that is : for all $n \in \mathbb{N}$ we have an isomorphism of abelian groups

$$H_n(X - U, A - U) \simeq H_n(X, A).$$

We now see how to use the Excision theorem with a first example.

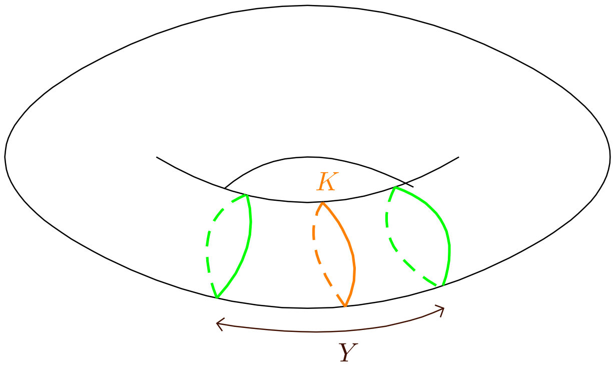
1.2.1. *Homology groups of the torus.*

Proposition 1.6. *The torus has homology groups:*

$$H_i(\mathbb{T}) \simeq \begin{cases} \mathbb{Z} & \text{if } i = 0 \\ \mathbb{Z} \oplus \mathbb{Z} & \text{if } i = 1 \\ \mathbb{Z} & \text{if } i = 2 \\ 0 & \text{else} \end{cases}$$

Proof. We will use the Excision theorem together with the associated long exact sequence of a pair to compute the homology groups of the torus. Let us parametrize $\mathbb{S}^1$ by means of the map $[0, 1] \longrightarrow \mathbb{S}^1; t \mapsto e^{2i\pi t}$ and consider the open subspace

$Y = \mathbb{S}^1 \times \left]\frac{1}{4}, \frac{3}{4}\right[$ and the closed subset $K = \mathbb{S}^1 \times \left\{\frac{1}{2}\right\}$ of $\mathbb{T} = \mathbb{S}^1 \times \mathbb{S}^1$.



Take now $A = Y \backslash K$ and $X = T \backslash K$. As we have $\bar{K} = K \subset \mathring{Y} = Y$, we can apply the Excision theorem to get an isomorphism $H_i(X, A) \simeq H_i(T, Y)$ for all $i \geqslant 0$.

The pair (X, A) induces a long exact sequence

$$\cdots 0 \longrightarrow H_2(X, A) \xrightarrow{\delta_1} H_1(A) \xrightarrow{i_*} H_1(X) \xrightarrow{k_*} H_1(X, A) \xrightarrow{\delta_0} \cdots$$

$$\cdots \longrightarrow H_0(A) \xrightarrow{j_*} H_0(X) \xrightarrow{l_*} H_0(X, A) \longrightarrow 0$$

Let us now compute the components of the sequence:

- First of all, A is homotopically equivalent to $\mathbb{S}^1 \sqcup \mathbb{S}^1$ and X is homotopically equivalent to $\mathbb{S}^1$ so we have $H_0(A) \simeq H_1(X) \simeq \mathbb{Z} \oplus \mathbb{Z}$, $H_0(X) \simeq H_1(X) \simeq \mathbb{Z}$;
- We have $H_2(X, A) \simeq \mathbb{Z}$ as δ_1 is injective and $\mathrm{Ker}(i_*) = \langle(-1, 1)\rangle \simeq \mathbb{Z}$;
- Next, $H_0(X, A) \simeq 0$ as j_* is surjective, $l_*(a) = 0$ for all $a \in \mathbb{Z}$. But l_* is also surjective so $H_0(X, A) \simeq \mathrm{im}(k_*) \simeq 0$;
- Finally, $H_1(X, A) \simeq \mathbb{Z}$ as i_* is surjective.

Now let us write the long exact sequence associated to the pair (T, Y).

$$\cdots 0 \longrightarrow H_2(\mathbb{T}) \longrightarrow H_2(T, Y) \xrightarrow{\alpha} H_1(Y) \xrightarrow{a_*} H_1(T) \xrightarrow{b_*} H_1(T, Y) \xrightarrow{\beta} \cdots$$

$$\cdots \longrightarrow H_0(Y) \xrightarrow{c_*} H_0(T) \xrightarrow{d_*} H_0(T, Y) \longrightarrow 0$$

- By Excision $H_2(T, Y) \simeq \mathbb{Z}$, $H_1(T, Y) \simeq \mathbb{Z}$ and $H_0(T, Y) \simeq 0$;
- We have $H_0(Y) \simeq H_1(Y) \simeq \mathbb{Z}$;
- The map α fits in the commutative square

$$\begin{array}{ccc} H_2(T\backslash K, Y\backslash K) & \xrightarrow{\delta_1} & H_1(Y\backslash K) \\ \downarrow & & \downarrow \lambda_* \\ H_2(T, Y) & \xrightarrow{\alpha} & H_1(Y) \end{array}$$

where $\lambda : Y\backslash K \longrightarrow Y$ is the inclusion map. Now, as $\lambda_*(a, b) = a + b$ and $\delta_1(1) = (-1, 1)$, we get that α is the zero map by commutativity of the square. A similar argument shows that β is also the zero map.

We get two induced exact sequences:

$$0 \longrightarrow H_0(Y) \longrightarrow H_0(T) \longrightarrow 0$$

and

$$0 \longrightarrow H_2(T) \longrightarrow H_2(T, Y) \longrightarrow 0$$

which force the isomorphisms $H_0(T) \simeq \mathbb{Z}$ and $H_2(T) \simeq \mathbb{Z}$. There is also an induced short exact sequence

$$0 \longrightarrow H_1(Y) \longrightarrow H_1(T) \longrightarrow H_1(T, Y) \longrightarrow 0.$$

As $H_1(T, Y) \simeq \mathbb{Z}$, this short exact sequence of abelian groups has a section and thus $H_1(T) \simeq \mathbb{Z} \oplus \mathbb{Z}$.

This concludes our proof. □

Remark 1.7. Let us try to compute the homology of this space using the Mayer-Vietoris sequence. We identify T with the revolution torus in the space $\mathbb{R}^3$ and we cut it into two open cylinders $C_1 \simeq \mathbb{S}^1 \times]0,1[$, $C_2 \simeq]0,1[\times \mathbb{S}^1$ with $C_1 \cap C_2 \simeq \mathbb{S}^1 \times \{1\} \sqcup \{1\} \times \mathbb{S}^1$ which has homology groups $H_i(\mathbb{S}^1) \oplus H_i(\mathbb{S}^1)$. The Mayer-Vietoris sequence is then:

$$0 \longrightarrow H_2(T) \longrightarrow H_1(C_1 \cap C_2) \xrightarrow{(H_1(j_1),H_1(j_2))} H_1(C_1) \oplus H_1(C_2) \longrightarrow \dots$$

$$\dots \longrightarrow H_1(T) \longrightarrow H_0(C_1 \cap C_2) \xrightarrow{(H_1(j_1),H_1(j_2))} H_0(C_1) \oplus H_0(C_2) \longrightarrow H_0(T) \longrightarrow 0$$

We know that $H_0(T) \simeq H_0(C_1) \simeq H_0(C_2) \simeq H_0(C_1 \cap C_2) \simeq \mathbb{Z}$ as $T, C_1, C_1 \cap C_2$ and C_2 are arcwise connected.

Let us show that $H_2(T) \simeq \mathbb{Z}$: we know that $H_2(T)$ is isomorphic to the kernel of $(H_1(j_1), H_1(j_2))$. We calculate $H_1(j_1)$ and $H_1(j_2)$ and see that they identify with the map

$$\begin{array}{ccc} \mathbb{Z}^2 & \longrightarrow & \mathbb{Z} \\ (x,y) & \longmapsto & x+y \end{array}$$

We conclude that $H_2(T) \simeq \mathbb{Z}$.

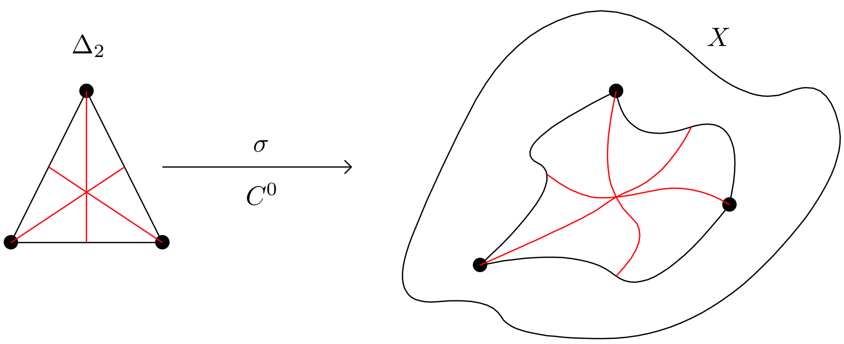
The problem now is that there is no evident way to compute $H_1(T)$, this is the reason why we used the Excision theorem in the above proof rather than the Mayer-Vietoris sequence, which works straightforwardly.

1.3. **The Small Chains Theorem and some consequences.** In the above subsections we didn't prove Mayer-Vietoris and Excision theorems because of the use of a very deep result about subdivision of chain complexes : the Small Chains Theorem.

Definition 1.8. Let X be a topological space and $\{U_\alpha\}_{\alpha \in I}$ be an open cover of X. The *small chains complex* $C_\bullet(X^{\{U_\alpha\}})$ *of* X is the sub-complex of $C_\bullet(X)$ such that $C_i(X^{\{U_\alpha\}})$ is the sub-module of $C_i(X)$ generated by the singular i-simplexes with image contained in one of the U_α.

Theorem 1.9. *The inclusion map* $C_\bullet(X^{\{U_\alpha\}}) \longrightarrow C_\bullet(X)$ *is a quasi-isomorphism i.e. it induces an isomorphism in homology groups.*

Proof. We will not prove this but the main tool in the proof is the notion of barycentric subdivision of a simplex, which can be illustrated as follows:



□

Remark 1.10. The Small chains theorem is at the heart of the proofs of the Mayer-Vietoris and Excision theorems. Indeed, up to some manipulations of modules and diagram chasing, we can prove Excision theorem by applying the small chains theorem to the cover $\mathcal{U} = (X - U, A)$ and we can prove the Mayer-Vietoris theorem by applying the small chains theorem to the cover $\mathcal{U} = (U, V)$. We redirect to [?Hatcher] for details about these proofs.

Let us now introduce a third homology computation tool which consists in deforming a space, in a homology-invariant way, into spaces which we know how to compute their homology groups.

Definition 1.11. Let A be a subset of a topological space X. The *collapse* of A in X is the following topological quotient

$$X/A := X/\sim$$

where $x \sim x' \iff x = x'$ or $x, x' \in A$.

This means that we collapsed all A into a point.

Example 1.12. For example the collapse $C(X)/X$ of X in the cone $C(X)$ is homeomorphic to the suspension $S(X)$ of X.

Exercise 1.13. Show that the restriction to $X - A$ of the projection $\pi : X \longrightarrow X/A$ is an homeomorphism into its image if A is open or closed.

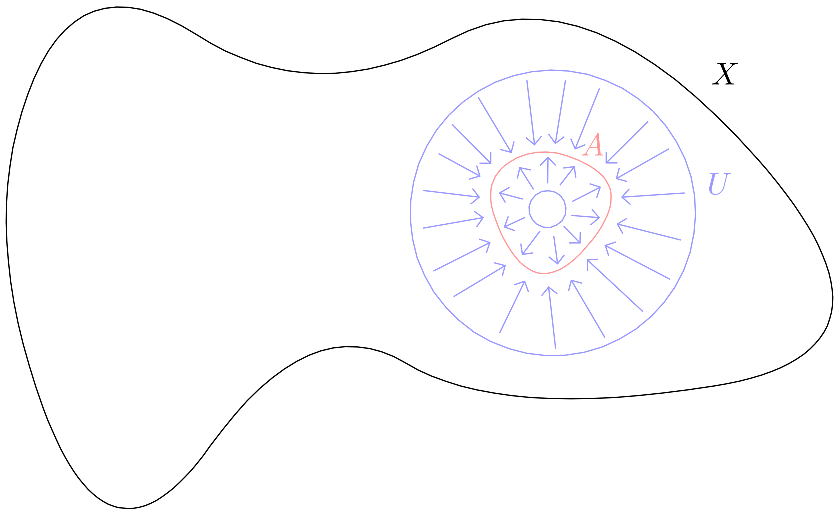
Definition 1.14. Let X be a topological space. Its reduced homology groups are, for all $i \geqslant 0$:

$$\widetilde{H}_i(X) := H_i(X, \{x\}).$$

Remark 1.15. We have explicitely $\widetilde{H}_i(X) = H_i(X)$ if $i > 0$ and $\widetilde{H}_0(X) \simeq \mathbb{Z}^{d-1}$ if $H_0(X) \simeq \mathbb{Z}^d$, for some $d \in \mathbb{N}$.

The following theorem relates the relative homology of a pair and the reduced homology of its associated collapse.

Theorem 1.16. *Let A be a closed subset of X which is a strong deformation retract of some open neighborhood.*



Then

$$\widetilde{H}_i(X/A) \simeq H_i(X, A).$$

Again, the proof of this result uses the Small Chains Theorem. We leave the reader the care to prove it. Now let us apply this tool to the computation of homology groups of a suspension of a space and of a bouquet of spaces.

Proposition 1.17. *Let X be a topological space. Then*

$$\widetilde{H}_i(S(X)) \simeq \widetilde{H}_{i-1}(X).$$

Proof. As $X \times \{0\} \subset C(X)$ is a deformation retract of a neighborhood, then by Theorem 1.16, we get a long exact sequence

$$\cdots \longrightarrow \widetilde{H}_i(C(X)) \longrightarrow \widetilde{H}_i(C(X)/X) \xrightarrow{g} \widetilde{H}_{i-1}(X) \longrightarrow \widetilde{H}_{i-1}(C(X)) \longrightarrow \cdots$$

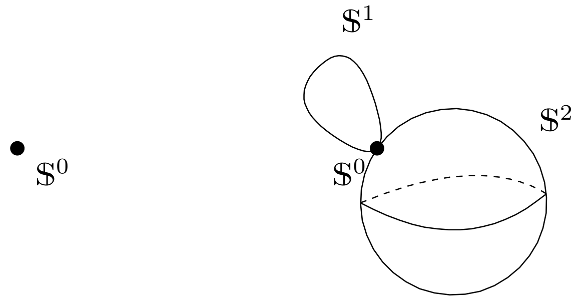
where $\widetilde{H}_i(C(X)) = \widetilde{H}_{i-1}(C(X)) \simeq 0$ as $C(X)$ is contractible and $C(X)/X$ is homeomorphic to $S(X)$. Then the exactness of the sequence implies that g is an isomorphism. $\square$

Definition 1.18. Let $(X_\alpha, x_\alpha)_{\alpha \in I}$ a countable family of pointed topological spaces. Then the *bouquet of* $(X_\alpha)_{\alpha \in I}$ is

$$\bigvee_{\alpha \in I} X_\alpha := \coprod_{\alpha \in I} X_\alpha / \sim$$

where $x \sim x'$ for all $\alpha, \beta \in I$.

For example, the bouquet $\bigvee_{k=0}^{2} \mathbb{S}^k$ of the spheres $\mathbb{S}^0$, $\mathbb{S}^1$ and $\mathbb{S}^2$ is



Proposition 1.19. *Let $(X_\alpha, x_\alpha)_{\alpha \in I}$ a countable family of pointed topological spaces. We suppose that x_α is a strong deformation retract of a neighborhood. Then we have*

$$\widetilde{H}_i\left(\bigvee_{\alpha \in I} X_\alpha\right) \simeq \bigoplus_{\alpha \in I} H_i(X_\alpha).$$

Proof. Set $X := \coprod_{\alpha \in I} X_\alpha$ and $A := \{x_\alpha; \alpha \in I\}$. Then $A \subset X$ is a deformation retract of a neighborhood and we get

$$H_i(X, A) \simeq \widetilde{H}_i(X/A) \simeq \bigoplus_{\alpha \in I} H_i(X_\alpha),$$

which concludes the proof. $\square$

Example 1.20. For all $n \in \mathbb{N}$, the bouquet $\bigvee_{k=0}^{n} \mathbb{S}^k$ of all spheres of dimension at most equal to n has reduced homology groups

$$\widetilde{H}_i\left(\bigvee_{k=0}^{n} \mathbb{S}^k\right) \simeq \begin{cases} \mathbb{Z} & \text{if } i = 0 \\ \mathbb{Z} & i = 1, \ldots, n \\ 0 & \text{else} \end{cases}.$$