Lesson n°5: Stochastic Differential Equations

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Introduction

Let $(B_t, t \geq 0)$ be a Brownian motion in $\mathbb{R}^d$. In this lesson, we are interested in solving stochastic differential equations of the form :

$$X_t = X_0 + \int_0^t \sigma(s, X_s)dB_s + \int_0^t b(s, X_s)ds$$

where X_0 is a given random variable, $b : \mathbb{R}^+ \times \mathbb{R}^n \longrightarrow \mathbb{R}$ is a measurable vector-valued function and $\sigma : \mathbb{R}^+ \times \mathbb{R}^n \longrightarrow \mathcal{M}_{n\times d}(\mathbb{R})$ is a measurable matrix-valued function. For any $i \in \{1, \dots, n\}$, we thus have componentwise :

$$X_t^{(i)} = X_0^{(i)} + \int_0^t \sum_{j=1}^d \sigma_{ij}(s, X_s)dB_s^{(j)} + \int_0^t b_i(s, X_s)ds.$$

Such an equation may also be written in a differential form:

$$\begin{cases} dX_t = \sigma(t, X_t)dB_t + b(t, X_t)dt, \\ X_0 \text{ a given random variable.} \end{cases}$$

We start with a few examples in dimension one.

Example 1.

i) (*Brownian motion with drift*) Take $X_0 = x \in \mathbb{R}$, $b \in \mathbb{R}$ and $\sigma = 1$. Then $X_t = x + B_t + bt$ is a Brownian motion with drift.

ii) (*Ornstein-Uhlenbeck process*) Consider the SDE :

$$\begin{cases} dX_t = dB_t - \lambda X_t dt, \\ X_0 = x. \end{cases}$$

The solution of this equation is given by $X_t = xe^{-\lambda t} + e^{-\lambda t}\int_0^t e^{\lambda s}dB_s$ and is called an Ornstein-Uhlenbeck process of parameter λ. X is a Gaussian process which has many applications in physics, when modeling for instance the displacement of particles in a fluid.

iii) (*Geometric Brownian motion*) Consider the SDE :

$$\begin{cases} dX_t = \sigma X_t dB_t + \mu X_t dt, \\ X_0 = x. \end{cases}$$

The solution of this equation is given by $X_t = x\exp\left(\sigma B_t + \left(\mu - \frac{\sigma^2}{2}\right)t\right)$ and is called a geometric Brownian motion. This process was used by Black, Merton and Scholes to model the prices of assets in their celebrated work on options pricing.

1 Definitions

To study stochastic differential equations, we must first define the notion of solution, as well as the notion of uniqueness. Let $(B_t, t \geq 0)$ be Brownian motion in $\mathbb{R}^d$ defined on a filtered space $(\Omega, \mathcal{F}, (\mathcal{F}_t,\, t \geq 0), \mathbb{P})$ and denote by $(\mathcal{F}_t^B,\, t \geq 0)$ its natural filtration completed with respect to $\mathbb{P}$. Consider the SDE

$$X_t = X_0 + \int_0^t \sigma(s, X_s) dB_s + \int_0^t b(s, X_s) ds. \tag{$*$}$$

1.1 Definition of solutions

Definition 2 (Solution)**.**
A solution of the stochastic differential equation $(*)$ is a pair (X, B) of adapted processes defined on a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t,\, t \geq 0), \mathbb{P})$ such that

i) B is a $(\mathcal{F}_t)$-Brownian motion in $\mathbb{R}^d$

ii) For every $1 \leq i \leq n$, $\int_0^t |b_i(s, X_s)| ds + \sum_{j=1}^d \int_0^t \sigma_{ij}^2(s, X_s) ds < +\infty$ a.s.

iii) The pair (X, B) satisfies Equation $(*)$ a.s.

Remark 3. In particular, a solution is a continuous semimartingale. Note also that Condition *ii)* may be replaced by b and σ are locally bounded.

Definition 4 (Strong and weak solution)**.**
A solution (X, B) of the stochastic differential equation $(*)$ is said to be strong *if X is adapted to the filtration $(\mathcal{F}_t^B)$. A solution which is not strong will be called* weak.

Example 5. Consider the SDE:

$$X_t = \int_0^t \operatorname{sgn}(X_s) dB_s.$$

By Lévy's characterization theorem, X is a Brownian motion and this SDE admits a weak solution. However, this solution is not strong. Indeed, assume that $\mathcal{F}_t^X \subset \mathcal{F}_t^B$. From Tanaka's formula

$$B_t = \int_0^t \operatorname{sgn}(X_s) dX_s = |X_t| - L_t^0(X)$$

so we deduce that $\mathcal{F}_t^B \subset \mathcal{F}_t^{|X|}$, hence $\mathcal{F}_t^X \subset \mathcal{F}_t^{|X|}$ which is a contradiction.

1.2 Uniqueness

Definition 6 (Uniqueness)**.**

1. *There is* pathwise uniqueness *for the stochastic differential equation (∗) if whenever* $(X^{(1)}, B)$ *and* $(X^{(2)}, B)$ *are two solutions defined on the same filtered probability space with respect to the same Brownian motion and* $X_0^{(1)} = X_0^{(2)}$ *a.s. then:*

$$\mathbb{P}\left(X_t^{(1)} = X_t^{(2)},\ \forall\, 0 \le t < +\infty\right) = 1.$$

2. *There is* uniqueness in law *for the stochastic differential equation (∗) if whenever* $(X^{(1)}, B^{(1)})$ *and* $(X^{(2)}, B^{(2)})$ *are two solutions, with possibly different Brownian motions, and* $X_0^{(1)} \overset{\text{(law)}}{=} X_0^{(2)}$*, then the laws of* $X^{(1)}$ *and* $X^{(2)}$ *are equal.*

Remark 7. It may be proven that pathwise uniqueness implies uniqueness in law, although it does not seem obvious from the definition.

Example 8. Consider the stochastic differential equation

$$X_t = X_0 + B_t + \int_0^t b(s, X_s)ds$$

where $(B_t, t \ge 0)$ is a standard Brownian motion and $b : \mathbb{R}^+ \times \mathbb{R} \longrightarrow \mathbb{R}$ is a measurable and bounded function which is decreasing in the space variable. Then pathwise uniqueness holds for this equation. Indeed, assume that $(X^{(1)}, B)$ and $(X^{(2)}, B)$ are two solutions with respect to the same Brownian motion and that $X_0^{(1)} = X_0^{(2)}$ a.s. By subtraction, we have

$$X_t^{(1)} - X_t^{(2)} = \int_0^t \left(b(s, X_s^{(1)}) - b(s, X_s^{(2)})\right) ds$$

hence the integration by parts formula yields :

$$(X_t^{(1)} - X_t^{(2)})^2 = 2\int_0^t (X_s^{(1)} - X_s^{(2)})d(X^{(1)} - X^{(2)})_t = 2\int_0^t (X_s^{(1)} - X_s^{(2)})\left(b(s, X_s^{(1)}) - b(s, X_s^{(2)})\right) ds \le 0$$

since b is decreasing in its space variable. Therefore, for any $t \ge 0$, $X_t^{(1)} = X_t^{(2)}$ a.s. and pathwise uniqueness follows since $X^{(1)}$ and $X^{(2)}$ are a.s. continuous.

Example 9. Consider (once again) the SDE:

$$X_t = \int_0^t \operatorname{sgn}(X_s)dB_s.$$

This SDE admits uniqueness in law since X is a Brownian motion. However, the pair $(-X, B)$ is also a weak solution, hence there is no pathwise uniqueness.

2 Existence and uniqueness of strong solutions

Let $(B_t, t \ge 0)$ be Brownian motion in $\mathbb{R}^d$ with natural filtration $(\mathcal{F}_t^B, t \ge 0)$ and consider the stochastic differential equation

$$\begin{cases} dX_t = \sigma(t, X_t)dB_t + b(t, X_t)dt, \\ X_0 \text{ a given random variable.} \end{cases}$$

where $b : \mathbb{R}^+ \times \mathbb{R}^n \longrightarrow \mathbb{R}$ is a measurable vector-valued function and $\sigma : \mathbb{R}^+ \times \mathbb{R}^n \longrightarrow \mathcal{M}_{n\times d}(\mathbb{R})$ is a measurable matrix-valued function. We consider the L^2 norms

$$\|b\|_2^2 = \sum_{i=1}^n b_i^2 \qquad \text{and} \qquad \|\sigma\|_2^2 = \sum_{i=1}^n \sum_{j=1}^d \sigma_{ij}^2.$$

Assume that X_0 is independent from $\mathcal{F}^B_\infty$ and define the augmented filtration

$$\mathcal{G}_t = \sigma(X_0) \vee \mathcal{F}^B_t = \sigma(X_0, B_t; 0 \leq s \leq t)$$

as well as the collection of null sets

$$\mathcal{N} = \{N \in \Omega,\ \exists A \in \mathcal{G}_\infty \text{ such that } N \subset A \text{ and } \mathbb{P}(A) = 0\}.$$

We finally set :

$$\mathcal{F}_t = \sigma(\mathcal{G}_t \cup \mathcal{N}).$$

$(\mathcal{F}_t, t \geq 0)$ is a filtration with satisfy the usual conditions (i.e. which is right-continuous and complete), and B remains a Brownian motion with respect to $(\mathcal{F}_t, t \geq 0)$.

Theorem 10. *Assume that the coefficients b and σ satisfy*

i) a global Lipschitz condition

$$\exists K > 0,\ \forall x, y \in \mathbb{R}^n, \qquad \|b(t,x) - b(t,y)\|_2 + \|\sigma(t,x) - \sigma(t,y)\|_2 \leq K\|x - y\|_2,$$

ii) and a linear growth condition

$$\exists K > 0,\ \forall x \in \mathbb{R}^n, \qquad \|b(t,x)\|_2^2 + \|\sigma(t,x)\|_2^2 \leq K^2(1 + \|x\|^2),$$

iii) and that X_0 is independent from B with $\mathbb{E}[\|X_0\|^2] < +\infty$.

Then, the SDE $(*)$ has a pathwise unique strong solution, and for any $T > 0$, there exists a constant C, which only depends on K and T, such that

$$\mathbb{E}\left[\|X_t\|^2\right] \leq C(1 + \mathbb{E}\left[\|X_0\|^2\right])e^{Ct} \qquad 0 \leq t \leq T.$$

The proof of this result is based on an approximation procedure,

$$\begin{cases} X_t^{(0)} = X_0 \\ X_t^{(k+1)} = X_0 + \displaystyle\int_0^t \sigma(s, X_s^{(k)})dB_s + \int_0^t b(s, X_s^{(k)})ds \end{cases}$$

by proving that the sequence of processes $X^{(k)}$ is a Cauchy sequence in the Banach space $\mathcal{C}([0,T], \mathbb{R})$ for any $T > 0$. The required estimates are obtained thanks to the isometry property of Itô's integral.

Remark 11. In particular, if b and σ do not depend on the time parameter, then the linear growth condition is a consequence of the global Lipschitz condition. Observe that the three examples given in the introduction fit in this theorem.

Example 12 (Strong existence without uniqueness)**.**
Consider the stochastic differential equation

$$X_t = 2\int_0^t X_s^{2/3} dB_s + 3\int_0^t X_s^{1/3} ds.$$

Then, for any $\theta \geq 0$, the process defined by :

$$X_t^{(\theta)} = \begin{cases} 0 & \text{if } 0 \leq t \leq T_0 = \inf\{s \geq \theta;\ B_s = 0\} \\ B_t^3 & \text{if } t \geq T_0 \end{cases}$$

is a strong solution of this SDE.

Definition 13 (Time homogeneous Itô diffusion).
Assume that the coefficients b and σ do not depend on the time parameter, and satisfy a global Lipschitz condition

$$\exists K > 0,\ \forall x, y \in \mathbb{R}^n, \qquad \|b(x) - b(y)\|_2 + \|\sigma(x) - \sigma(y)\|_2 \leq K\|x - y\|_2.$$

The pathwise unique strong solution of the SDE:

$$dX_t = \sigma(X_t)dB_t + b(X_t)dt, \qquad X_0 = x \in \mathbb{R},$$

is called an (time-homogeneous) Itô diffusion.

Theorem 14 (Strong Markov property).
An Itô diffusion enjoys the strong Markov property: for any positive $\mathcal{F}_\infty$-measurable random variable Z and any a.s. finite stopping time T with respect to $(\mathcal{F}_t, t \geq 0)$ we have

$$\mathbb{E}_x \left[Z \circ \theta_T | \mathcal{F}_T\right] = \mathbb{E}_{X_T} \left[Z\right]$$

where θ denotes the usual translation operator.

We shall now look for weak solutions.

3 Girsanov's theorem

One way to prove the existence of a weak solution to a stochastic differential equation is to use an absolutely continuous change of probability measures. This is the purpose of Girsanov's theorem, which states that if $\mathbb{Q}$ is a probability measure on $(\Omega, \mathcal{F})$ which is absolutely continuous with respect to $\mathbb{P}$, then every semimartingale with respect to $\mathbb{P}$ remains a semimartingale with respect to $\mathbb{Q}$.

3.1 Change of probability

Let $(\Omega, \mathcal{F}, (\mathcal{F}_t, t \geq 0), \mathbb{P})$ be a filtered probability space. We assume that $(\mathcal{F}_t)$ is a right-continuous and complete with terminal σ-field $\mathcal{F}_\infty$.

Definition 15. *Let $\mathbb{Q}$ be a probability measure on $(\Omega, \mathcal{F})$.*

i) *$\mathbb{Q}$ is said to be weakly absolutely continuous with respect to $\mathbb{P}$ (we denote $\mathbb{Q} \lhd \mathbb{P}$) is there exists a positive process $(D_t, t \geq 0)$ such that*
$$\mathbb{Q}_{|\mathcal{F}_t} = D_t \bullet \mathbb{P}_{|\mathcal{F}_t}.$$

ii) *$\mathbb{Q}$ is absolutely continuous with respect to $\mathbb{P}$ (we denote $\mathbb{Q} \ll \mathbb{P}$) if there exists a random variable D_∞ such that*
$$\mathbb{Q} = D_\infty \bullet \mathbb{P}.$$

Remark 16. Observe that the process $(D_t, t \geq 0)$ is necessarily a positive martingale. Indeed, let $s < t$ and $A \in \mathcal{F}_s$. Since A is also $\mathcal{F}_t$-measurable, we have $\mathbb{Q}_{\mathcal{F}_t}(A) = \mathbb{Q}_{|\mathcal{F}_s}(A)$ and the weak absolute continuity formula gives $\mathbb{E}[D_t 1_A] = \mathbb{E}[D_s 1_A]$ which proves that D is a positive $((\mathcal{F}_t), \mathbb{P})$-martingale. As such, D converges a.s. towards a random variable D_∞ and from Fatou's lemma

$$\mathbb{E}[D_\infty] = \mathbb{E}\left[\lim_{t \to +\infty} D_t\right] \leq \liminf_{t \to +\infty} \mathbb{E}[D_t] = 1$$

since $\mathbb{E}[D_t] = \mathbb{Q}_{|\mathcal{F}_t}(\Omega) = 1$. If the convergence of D also holds in L^1, we have the equivalences :

$$(D_t, t \geq 0) \text{ is a uniformly integrable martingale} \iff \mathbb{E}[D_\infty] = 1 \iff \mathbb{Q} \ll \mathbb{P}.$$

Proposition 17. *Assume that $\mathbb{Q} \lhd \mathbb{P}$. Then*

i) if T is a $(\mathcal{F}_t)$-stopping time, then

$$\forall A \in \mathcal{F}_T \cap \{T < +\infty\}, \qquad \mathbb{Q}(A) = \int_A D_T \centerdot d\,\mathbb{P}.$$

If $\mathbb{Q} = D_\infty \centerdot \mathbb{P}$, this relation is valid for any $A \in \mathcal{F}_T$.

ii) The martingale D is strictly positive $\mathbb{Q}$-a.s., i.e. $\mathbb{Q}(D_t > 0, \forall t \geq 0) = 1$.

Proof. If T is a $(\mathcal{F}_t)$-stopping time, for any $A \in \mathcal{F}_T$

$$\mathbb{Q}(A \cap \{T \leq t\}) = \int_{A\cap\{T\leq t\}} D_t \, d\,\mathbb{P} = \int_{A\cap\{T\leq t\}} D_{t\wedge T} \, d\,\mathbb{P}$$

and letting $t \to +\infty$

$$\mathbb{Q}(A \cap \{T < +\infty\}) = \int_{A\cap\{T<+\infty\}} D_T \, d\,\mathbb{P}.$$

If $\mathbb{Q} = D_\infty \centerdot \mathbb{P}$, this relation is also valid when $T = +\infty$. Point *ii)* is then a consequence of Point *i)* applied with the stopping time $T_0 = \inf\{t \geq 0,\, D_t = 0\}$:

$$\mathbb{Q}(T < +\infty) = \int_{\{T<+\infty\}} D_T \, d\,\mathbb{P} = 0.$$

■

Theorem 18 (Girsanov's theorem)**.**
Assume that $\mathbb{Q} \lhd \mathbb{P}$. Let $(M_t,\, t \geq 0)$ be a continuous $((\mathcal{F}_t), \mathbb{P})$-local martingale. Then, the process

$$\widetilde{M_t} = M_t - \int_0^t \frac{1}{D_s} d\langle M, D\rangle_s$$

is a continuous $((\mathcal{F}_t), \mathbb{Q})$-local martingale.

When the continuous martingale D is also strictly positive $\mathbb{P}$-a.s., we may obtain another formulation.

Corollary 19. *Assume that $\mathbb{Q} \lhd \mathbb{P}$ and that D is strictly positive $\mathbb{P}$-a.s.*

i) There exists a unique continuous local martingale L such that

$$D = \exp\left(L - \frac{1}{2}\langle L, L\rangle\right).$$

ii) Let $(M_t,\, t \geq 0)$ be a continuous $((\mathcal{F}_t), \mathbb{P})$-local martingale. Then, the process

$$\widetilde{M_t} = M_t - \langle L, M\rangle_t = M_t - \int_0^t \frac{1}{D_s} d\langle D, M\rangle_s$$

is a continuous $((\mathcal{F}_t), \mathbb{Q})$-local martingale.

Proof. Assume first that there are two continuous local martingales $L^{(1)}$ and $L^{(2)}$ such that

$$\exp\left(L^{(1)} - \frac{1}{2}\langle L^{(1)}, L^{(1)}\rangle\right) = \exp\left(L^{(2)} - \frac{1}{2}\langle L^{(2)}, L^{(2)}\rangle\right).$$

This implies that $L^{(1)} - L^{(2)} = \frac{1}{2}\langle L^{(1)}, L^{(1)}\rangle - \frac{1}{2}\langle L^{(2)}, L^{(2)}\rangle$ is a continuous local martingale with finite variation, hence $L^{(1)} = L^{(2)}$ a.s. The result then follows by applying Itô's formula to $\ln(D_t)$:

$$\begin{aligned}\ln(D_t) &= \ln(D_0) + \int_0^t \frac{1}{D_s} dD_s - \frac{1}{2}\int_0^t \frac{1}{D_s^2} d\langle D, D\rangle_t \\ &= \ln(D_0) + L_t - \frac{1}{2}\langle L, L\rangle_t\end{aligned}$$

and noticing that

$$\langle M, L\rangle_t = \left\langle \int_0^\cdot dM_s, \int_0^\cdot \frac{1}{D_s} dD_s \right\rangle_t = \int_0^t \frac{1}{D_s}\langle M, D\rangle_s.$$

■

Corollary 20. *Assume that $\mathbb{Q} \lhd \mathbb{P}$ and that $D = \exp(L - \frac{1}{2}\langle L, L\rangle)$ is a continuous martingale. Let $(B_t,\, t \geq 0)$ be a $((\mathcal{F}_t), \mathbb{P})$-Brownian motion. Then, the process*

$$\widetilde{B}_t = B_t - \langle L, B\rangle_t = B_t - \int_0^t \frac{1}{D_s} d\langle D, B\rangle_s$$

is a $((\mathcal{F}_t), \mathbb{Q})$-Brownian motion.

Proof. This is a direct consequence of Lévy characterization of Brownian motion.

■

3.2 Application to SDE

Let L be a continuous local martingale. In general, from Itô's formula, the process $D = \exp\left(L - \frac{1}{2}\langle L, L\rangle\right)$ is only a continuous local martingale, hence, to define a new measure $\mathbb{Q}$ by a weak absolute continuity formula, we need to know if D is actually a true martingale. A sufficiency condition for this is given by the following Novikov's condition.

Theorem 21 (Novikov's condition)**.**
Let L be a continuous local martingale and define $D = \exp\left(L - \frac{1}{2}\langle L, L\rangle\right)$. If

$$\mathbb{E}\left[\exp\left(\frac{1}{2}\langle L, L\rangle_t\right)\right] < +\infty, \qquad \forall t \geq 0,$$

then D is a true continuous martingale.

This result allows to construct weak solutions.

Theorem 22. *Consider a fixed horizon $T > 0$ and let $b : [0, T] \times \mathbb{R}^n \longrightarrow \mathbb{R}^n$ be a measurable function such that*

$$\|b(t, x)\|_2 \leq K(1 + \|x\|_2) \qquad \forall\, 0 \leq t \leq T,\ x \in \mathbb{R}^n$$

for a constant $K > 0$. Then, for every initial condition $x \in \mathbb{R}$, the stochastic differential equation

$$dX_t = dB_t + b(t, X_t)dt$$

admits a weak solution.

Sketch of proof. Let $(X_t, t \geq 0)$ be a n-dimensional Brownian motion defined on a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t), \mathbb{P})$ and started at $x \in \mathbb{R}^n$. Consider the process

$$D_t = \exp\left(\sum_{i=1}^n \int_0^t b_i(s, X_s)dX_s - \frac{1}{2}\int_0^t \|b(s, X_s)\|_2^2 ds\right).$$

It may be prove thanks to a (slight generalization of) Novikov's condition that D is actually a true martingale. Therefore, the process

$$B_t = X_t - x - \int_0^t b(s, X_s)ds$$

is a $((\mathcal{F}_t), \mathbb{Q})$-Brownian motion hence, the process (X, B) defined on the filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t), \mathbb{Q})$ is a weak solution with initial condition $X_0 = x$ a.s.

■

4 Feynman-Kac formulae

We shall now establish some links between Itô diffusions and partial differential equations.

Definition 23 (Infinitesimal generator)**.**
Let X be an Itô diffusion in $\mathbb{R}^n$. The infinitesimal generator of X is defined by

$$Af(x) = \lim_{t \downarrow 0} \frac{\mathbb{E}_x[f(X_t)] - f(x)}{t}, \qquad x \in \mathbb{R}^n.$$

The set of functions such that the limit exists for any $x \in \mathbb{R}^n$ is denoted by $\mathcal{D}_A$.

Theorem 24. *Let X be an Itô diffusion in $\mathbb{R}^n$. If $f \in \mathcal{C}^2(\mathbb{R}^n)$ with compact support, then $f \in \mathcal{D}_A$ and*

$$Af(x) = \frac{1}{2} \sum_{i,j=1}^n a_{i,j}(x) \frac{\partial^2 f}{\partial x_i \partial x_j}(x) + \sum_{i=1}^n b_i(x) \frac{\partial f}{\partial x_i}(x) \qquad \text{where } a_{i,j}(x) = (\sigma(x)\sigma^*(x))_{ij}.$$

Furthermore, for any stopping time τ with finite expectation $\mathbb{E}[\tau] < +\infty$ and $f \in \mathcal{C}^2(\mathbb{R}^n)$ with compact support, we have Dynkin's formula:

$$\mathbb{E}_x\left[f(X_\tau)\right] = f(x) + \mathbb{E}_x\left[\int_0^\tau Af(X_s)ds\right].$$

Sketch of Proof. The proof relies plainly on Itô's formula. Indeed, with obvious notation :

$$\mathbb{E}_x\left[f\left(X_t^{(1)}, \dots, X_t^{(n)}\right)\right] = f(x) + \mathbb{E}_x\left[\sum_{i=1}^n \int_0^t \frac{\partial f}{\partial x_i}(X_s)dX_s^{(i)} + \frac{1}{2}\sum_{i=1}^n \sum_{j=1}^n \int_0^t \frac{\partial^2 f}{\partial x_i \partial x_j}(X_s)d\langle X^{(i)}, X^{(j)}\rangle_s\right].$$

Now, the bracket between $X^{(i)}$ and $X^{(j)}$ equals :

$$\begin{aligned}
\langle X^{(i)}, X^{(j)}\rangle_t &= \left\langle \sum_{k=1}^n \int_0^{\cdot} \sigma_{ik}(X_s)dB_s^{(k)}, \sum_{k=1}^n \int_0^{\cdot} \sigma_{jk}(X_s)dB_s^{(k)} \right\rangle_t \\
&= \sum_{k=1}^n \int_0^t \sigma_{ik}(X_s)\sigma_{jk}(X_s)ds \\
&= \sum_{k=1}^n \int_0^t \sigma_{ik}(X_s)\sigma_{kj}^*(X_s)ds \\
&= \int_0^t (\sigma(X_s)\sigma^*(X_s))_{ij}ds.
\end{aligned}$$

Using the fact that the stochastic integrals with respect to Brownian motions are martingales with null expectation, we obtain:

$$\mathbb{E}_x[f(X_t)] - f(x) = \mathbb{E}_x\left[\sum_{i=1}^n \int_0^t \frac{\partial f}{\partial x_i}(X_s)b_i(X_s)ds + \frac{1}{2}\sum_{i=1}^n \sum_{j=1}^n \int_0^t \frac{\partial^2 f}{\partial x_i \partial x_j}(X_s)(\sigma(X_s)\sigma^*(X_s))_{ij}ds\right].$$

This is Dynkin's formula with the constant stopping time $\tau = t$. It remains to divide this expression by t and to let $t \to 0$ to obtain the infinitesimal generator A.

■

A useful relationship between the Itô diffusion X and its infinitesimal generator is presented in the following Feynman-Kac formulae.

Theorem 25 (The Feynman-Kac formula)**.**
Let $f \in \mathcal{C}^2(\mathbb{R}^n)$ with compact support and $V : \mathbb{R}^n \longrightarrow \mathbb{R}^+$ be a continuous function. Define the function

$$u(t,x) = \mathbb{E}_x\left[e^{-\int_0^t V(X_s)ds} f(X_t)\right].$$

Then, u solves the Cauchy problem

$$\begin{cases} \dfrac{\partial u}{\partial t}(t,x) + V(x)u(t,x) = Au(t,x) & (t,x) \in [0,+\infty[\times\mathbb{R} \\ u(0,x) = f(x). \end{cases}$$

Proof. Fix $t > 0$. For $h > 0$ we have, applying the Markov property :

$$\begin{aligned} \frac{1}{h}\left(\mathbb{E}_x\left[u(t,X_h)\right] - u(t,x)\right) &= \frac{1}{h}\left(\mathbb{E}_x\left[\mathbb{E}_{X_h}\left[e^{-\int_0^t V(X_s)ds} f(X_t)\right]\right] - \mathbb{E}_x\left[e^{-\int_0^t V(X_s)ds} f(X_t)\right]\right) \\ &= \frac{1}{h}\left(\mathbb{E}_x\left[\mathbb{E}_x\left[e^{-\int_0^t V(X_{s+h})ds} f(X_{t+h})|\mathcal{F}_h\right]\right] - \mathbb{E}_x\left[e^{-\int_0^t V(X_s)ds} f(X_t)\right]\right) \\ &= \frac{1}{h}\mathbb{E}_x\left[e^{-\int_h^{t+h} V(X_s)ds} f(X_{t+h}) - e^{-\int_0^t V(X_s)ds} f(X_t)\right]. \end{aligned}$$

This last term may be decomposed in

$$\frac{1}{h}\mathbb{E}_x\left[e^{-\int_0^{t+h} V(X_s)ds} f(X_{t+h}) - e^{-\int_0^t V(X_s)ds} f(X_t)\right] + \frac{1}{h}\mathbb{E}_x\left[e^{-\int_0^{t+h} V(X_s)ds} f(X_{t+h})\left(e^{\int_0^h V(X_s)ds} - 1\right)\right].$$

Now, the first term converges towards

$$\frac{1}{h}\mathbb{E}_x\left[e^{-\int_0^{t+h} V(X_s)ds} f(X_{t+h}) - e^{-\int_0^t V(X_s)ds} f(X_t)\right] = \frac{u(t+h,x) - u(t,x)}{h} \underset{h\to 0}{\longrightarrow} \frac{\partial u}{\partial t}(t,x)$$

while the second term

$$\frac{1}{h}\mathbb{E}_x\left[e^{-\int_0^{t+h} V(X_s)ds} f(X_{t+h})\left(e^{\int_0^h V(X_s)ds} - 1\right)\right] \underset{h\to 0}{\longrightarrow} u(t,x)V(x),$$

which ends the proof.

■

We may also state a converse version.

Theorem 26. *Let $f \in \mathcal{C}^2(\mathbb{R}^n)$ with compact support and $V : \mathbb{R}^n \longrightarrow \mathbb{R}^+$ be a continuous function. Let $u : [0,+\infty[\times\mathbb{R}^n \longrightarrow \mathbb{R}$ be a $\mathcal{C}^{1,2}(\mathbb{R}\times\mathbb{R}^n)$ function which is solution of the Cauchy problem*

$$\begin{cases} \dfrac{\partial u}{\partial t}(t,x) + V(x)u(t,x) = Au(t,x) \\ u(0,x) = f(x). \end{cases}$$

Assume furthermore that for each compact $K \subset \mathbb{R}$, the function u is bounded on $K \times \mathbb{R}^n$. Then,

$$u(t,x) = \mathbb{E}_x\left[e^{-\int_0^t V(X_s)ds} f(X_t)\right].$$

We now state and prove a simple one-dimensional version for Brownian motion, which relies on ordinary differential equations.

Theorem 27. *Let $V(x) \geq 0$ be a positive continuous function on $\mathbb{R}$, $\lambda > 0$, and let ϕ_+ and ϕ_- be two $\mathcal{C}^2$-solutions of the differential equation*

$$\frac{1}{2}\phi''(x) = (V(x)+\lambda)\phi(x) \tag{1}$$

such that :

$$\phi_+ \text{ is bounded on } [0,+\infty[\qquad \text{and} \qquad \phi_- \text{ is bounded on }]-\infty, 0].$$

Let $w_\lambda := \phi_+(0)\phi'_-(0) - \phi_-(0)\phi'_+(0)$ and assume that $\omega_\lambda \neq 0$. Then, for any bounded and measurable function f on $\mathbb{R}$ and any $x \in \mathbb{R}$:

$$\int_0^{+\infty} e^{-\lambda t}\mathbb{E}_x\left[e^{-\int_0^t V(B_s)ds} f(B_t)\right]dt = \frac{2}{\omega_\lambda}\left(\phi_+(x)\int_{-\infty}^x \phi_-(y)f(y)dy + \phi_-(x)\int_x^{+\infty}\phi_+(y)f(y)dy\right). \tag{2}$$

Proof. Define the Wronskien:

$$W_\lambda(x) = \phi_+(x)\phi'_-(x) - \phi_-(x)\phi'_+(x).$$

Since ϕ_+ and ϕ_- are solutions of (1), we deduce that $W'_\lambda(x) = 0$, hence for any $x \in \mathbb{R}$, $W_\lambda(x) = W_\lambda(0) = \omega_\lambda$. Assume first that f is continuous and has compact support, and define:

$$\phi(x) = \phi_+(x)\int_{-\infty}^x \phi_-(y)f(y)dy + \phi_-(x)\int_x^{+\infty}\phi_+(y)f(y)dy.$$

ϕ is a function of $\mathcal{C}^1$-class, and differentiation yields:

$$\phi'(x) = \phi'_+(x)\int_{-\infty}^x \phi_-(y)f(y)dy + \phi'_-(x)\int_x^{+\infty}\phi_+(y)f(y)dy.$$

We thus deduce that ϕ is actually of $\mathcal{C}^2$-class, and from (1):

$$\phi''(x) = 2(V(x)+\lambda)\phi(x) - W_\lambda(x)f(x) = 2(V(x)+\lambda)\phi(x) - \omega_\lambda f(x).$$

Observe also that, since f is a function with compact support, the function ϕ is bounded on $\mathbb{R}$. Consider now the process

$$M_t = e^{-\lambda t - \int_0^t V(B_s)ds}\phi(B_t) + \frac{\omega_\lambda}{2}\int_0^t e^{-\lambda u - \int_0^u V(B_s)ds} f(B_u)du.$$

From Itô's formula, this process is a local martingale and we have the estimate:

$$|M_t| \leq \sup_{x\in\mathbb{R}}|\phi(x)| + \frac{\omega_\lambda}{2}\sup_{x\in\mathbb{R}}|f(x)|\int_0^t e^{-\lambda u}du \leq \sup_{x\in\mathbb{R}}|\phi(x)| + \frac{\omega_\lambda}{2\lambda}\sup_{x\in\mathbb{R}}|f(x)|.$$

Therefore M is uniformly bounded, i.e. M is a bounded martingale and

$$\phi(x) = \mathbb{E}_x[M_0] = \mathbb{E}_x[M_\infty] = \frac{\omega_\lambda}{2}\mathbb{E}_x\left[\int_0^{+\infty} e^{-\lambda u - \int_0^u V(B_s)ds} f(B_u)du\right].$$

By a monotone class argument, the assumption on the continuity of f may be dropped, so Relation (2) is in fact valid for any positive and measurable function with compact support in $[-N,N]$. Choose now a real x such that $\phi_-(x) \neq 0$ and define $f_N(y) = \operatorname{sgn}(\phi_+(y))1_{\{x<y<K\}}$. Applying Formula (2) to f_N and letting $N \to +\infty$, we deduce that $\int_x^{+\infty}|\phi_+(y)|dy < +\infty$. Therefore, ϕ_+ is integrable at $+\infty$, and by a similar argument, so is ϕ_- at $-\infty$. The result then follows by dominated convergence, with the sequence of functions $f_N(x) = f(x)1_{\{|x|<N\}}$.

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Example 28. Let us apply this result to prove the third Arcsine law. Let $(B_s,\, s \geq 0)$ be a standard Brownian motion started from 0 and let

$$A_t = \int_0^t 1_{\{B_s>0\}} ds$$

be the time spent by Brownian motion over 0 in the time interval $[0,t]$. From the above Feynman-Kac formula :

$$\int_0^{+\infty} e^{-\lambda t}\mathbb{E}_0\left[e^{-a\int_0^t 1_{\{B_s>0\}}ds}\right]dt = \frac{2}{\omega_\lambda}\left(\phi_+(0)\int_{-\infty}^0 \phi_-(y)dy + \phi_-(0)\int_0^{+\infty}\phi_+(y)dy\right)$$

where ϕ_+ and ϕ_- are the appropriate solutions of the equation

$$\frac{1}{2}\phi''(x) = \left(a1_{\{x>0\}} + \lambda\right)\phi(x).$$

It is easily seen that we may choose

$$\begin{cases} \phi_+(x) = \exp\left(-x\sqrt{2a+2\lambda}\right) & x \geq 0 \\ \phi_-(x) = \exp\left(x\sqrt{2\lambda}\right) & x \leq 0 \end{cases}$$

so that

$$\begin{aligned}
\int_0^{+\infty} e^{-\lambda t}\mathbb{E}_0\left[e^{-a\int_0^t 1_{\{B_s>0\}}ds}\right]dt &= \frac{2}{\sqrt{2a+2\lambda}+\sqrt{2\lambda}}\left(\int_{-\infty}^0 e^{y\sqrt{2\lambda}}dy + \int_0^{+\infty} e^{-y\sqrt{2a+2\lambda}}dy\right) \\
&= \frac{2}{\sqrt{2a+2\lambda}+\sqrt{2\lambda}}\left(\frac{1}{\sqrt{2\lambda}} + \frac{1}{\sqrt{2a+2\lambda}}\right) \\
&= \frac{1}{\sqrt{\lambda}\sqrt{\lambda+a}} \\
&= \left(\int_0^{+\infty} e^{-\lambda t}\frac{dt}{\sqrt{\pi t}}\right)\left(\int_0^{+\infty} e^{-\lambda t}e^{-at}\frac{dt}{\sqrt{\pi t}}\right) \\
&= \int_0^{+\infty} e^{-\lambda t}\int_0^t e^{-as}\frac{ds}{\pi\sqrt{s}\sqrt{t-s}},
\end{aligned}$$

and the injectiveness of the Laplace transforms finally yields

$$\mathbb{E}_0\left[e^{-a\int_0^t 1_{\{B_s>0\}}ds}\right] = \int_0^{+\infty} e^{-as}\frac{1}{\pi\sqrt{s}\sqrt{t-s}}1_{\{0<s<t\}}\,ds.$$

5 Simulations

Choose a fixed horizon $T>0$ and, for $b:\mathbb{R}\longrightarrow\mathbb{R}$ and $\sigma:\mathbb{R}\longrightarrow\mathbb{R}^+$, consider the stochastic differential equation

$$dX_t = \sigma(X_t)dB_t + b(X_t)dt \tag{3}$$

where B is a standard Brownian motion. Let $n\in\mathbb{N}^*$ and set $\Delta t = \frac{T}{n}$ as discretization step.

Definition 29 (Discrete Euler Scheme)**.** *The discrete Euler scheme associated with the SDE (3) is given, for $0\leq k\leq n-1$, by :*

$$\begin{cases} X_0^{(n)} = X_0 \\ X_{(k+1)\Delta t}^{(n)} = X_{k\Delta t}^{(n)} + \sigma\left(X_{k\Delta t}^{(n)}\right)\left(B_{(k+1)\Delta t} - B_{k\Delta t}\right) + b\left(X_{k\Delta t}^{(n)}\right)\Delta t \end{cases}$$

Remark 30. The second relation may be written

$$X_{(k+1)\Delta t}^{(n)} = X_{k\Delta t}^{(n)} + \sigma\left(X_{k\Delta t}^{(n)}\right)\sqrt{\Delta t}G + b\left(X_{k\Delta t}^{(n)}\right)\Delta t$$

where G is a standard Gaussian random variable. Therefore, to simulate the discrete Euler scheme associated with the SDE (3) we only need to simulate n independent standard Gaussian random variables.

It is then natural to extend this discrete scheme to a continuous one by interpolation.

Definition 31 (Continuous Euler Scheme)**.** *The continuous Euler scheme associated with the SDE (3) is given by :*

$$\begin{cases} X_0^{(n)} & = X_0 \\ X_t^{(n)} & = X_{k\Delta t}^{(n)} + \sigma\left(X_{k\Delta t}^{(n)}\right)(B_t - B_{k\Delta t}) + b\left(X_{k\Delta t}^{(n)}\right)(t - k\Delta t) \qquad \text{for } t \in [k\Delta t, (k+1)\Delta t[\end{cases}$$

The following result states that the continuous Euler scheme converges towards the solution of (3) in any L^p.

Theorem 32. *Assume that b and σ are globally Lipschitz continuous and that $\mathbb{E}[|X_0|^{2p}] < +\infty$ for some $p \in \mathbb{N}^*$. Then, there exists a constant C_T such that*

$$\mathbb{E}\left[\sup_{0\le t\le T}\left|X_t - X_t^{(n)}\right|^{2p}\right] \le \frac{C_T}{n^p}.$$