Improved Sobolev inequalities: the case $p = 1$ and generalizations to classical Lorentz spaces

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May 7, 2015

Abstract

We give a general treatment of refined Sobolev inequalities in the case $p = 1$ and when $p > 1$ we study these inequalities using as base space classical Lorentz spaces associated to a weight from the Ariño-Muckenhoupt class B_p. The arguments used for the case $p = 1$ rely essentially on spectral theory while the ideas behind the case $p > 1$ are based on pointwise estimates and on the boundedness of Hardy-Littlewood maximal function. As a by-product we will also consider Morrey-Sobolev inequalities. This arguments can be generalized to many different frameworks, in particular the proofs are given in the setting of stratified Lie groups.

Keywords: Improved Sobolev inequalities, Sobolev spaces, Besov spaces, Classical Lorentz spaces, Stratified Lie groups.
Mathematics Subject Classification 46E35 ; 26D10 ; 46E30 ; 22E30

1 Introduction and presentation of the results

The aim of this article is to provide a general proof for improved Sobolev inequalities in the particular case when the parameter that defines the Sobolev space $\dot{W}^{s,p}$ in the right-hand side of the inequality satisfies $p = 1$ and to give some generalizations to Lorentz-Sobolev spaces in the case $p > 1$. These inequalities are of the following general form

$$\|f\|_{\dot{W}^{s_1,q}} \leq C\|f\|^{\theta}_{\dot{W}^{s,p}}\|f\|^{1-\theta}_{\dot{B}^{-\beta,\infty}_{\infty}}, \tag{1}$$

where $f : \mathbb{R}^n \longrightarrow \mathbb{R}$ is a function such that $f \in \dot{W}^{s,p} \cap \dot{B}^{-\beta,\infty}_{\infty}$. Here we write $\dot{W}^{s,p}$ for homogeneous Sobolev spaces and $\dot{B}^{-\beta,\infty}_{\infty}$ for homogeneous Besov spaces (see Section 4 below for precise definitions).

The parameters s, s_1, p, q and β defining Sobolev and Besov spaces in the previous inequality are related by the conditions $1 < p < q < +\infty$, $\theta = p/q$, $s_1 = \theta s - (1-\theta)\beta$ and $-\beta < s_1 < s$, but they do not depend on the dimension and in this sense these inequalities are more general than classical Sobolev inequalities; of course the inequalities above are sharper than classical ones.

It is worth noting that even for classical Sobolev inequalities, there are two types of proofs following the value of the index p of the L^p norm defining the Sobolev space $\dot{W}^{s,p}$ in the right-hand side of (1). Indeed, if $1 < p < +\infty$, the Sobolev space $\dot{W}^{1,p}$ can be defined by several and equivalent characterizations, but we do not have the same freedom for the Sobolev space $\dot{W}^{1,1}$. See [35] for a discussion in the case of classical Sobolev inequalities.

This issue concerning Sobolev spaces remains when considering improved Sobolev inequalities and it is also necessary to distinguish the case when $p > 1$ from the case when $p = 1$. Historically, the first proof of these inequalities is due to P. Gérard, F. Oru and Y. Meyer [21] and is based on a Littlewood-Paley decomposition and interpolation

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results applied to dyadic blocks. The inequalities obtained are of the form of (1) above, but it is very important to stress that the value $p = 1$ is forbidden here as the L^1 space does not admit a characterization via the Littlewood-Paley analysis. Another proof of these inequalities using maximal function and Hedberg's inequality is given in [11], but as maximal function is not bounded in L^1 it is not possible to apply this argument to the case of the Sobolev space $\dot{W}^{1,1}$.

A second method, studied by M. Ledoux in [25], use semi-group properties related to the Laplacian and it's associated heat kernel and allows us to treat the case $p = 1$. Indeed, if $\nabla f \in L^p(\mathbb{R}^n)$ and $f \in \dot{B}^{-\beta,\infty}_{\infty}(\mathbb{R}^n)$, we have

$$\|f\|_{L^q} \leq C\|\nabla f\|^{\theta}_{L^p}\|f\|^{1-\theta}_{\dot{B}^{-\beta,\infty}_{\infty}}, \tag{2}$$

with $1 \leq p < q < +\infty$, $\theta = p/q$ and $\beta = \theta/(1-\theta)$. Here we can consider the Sobolev space $\dot{W}^{1,1}$ in the right-hand side of the previous inequality, but is not possible to consider a Sobolev space $\dot{W}^{s_1,q}$ in the left-hand side as the proof relies in a cut-off argument which is not well suited for fractional Sobolev spaces. Another proof of this inequality based on non-increasing rearrangements functions was given by J. Martín and M. Milman [28], but the important case $p = 1$ can not be treated by this method.

A different method was proposed by A. Cohen, W. Dahmen, I. Daubechies & R. De Vore in [16] where these authors use a BV-norm weak estimation using wavelet coefficients and isoperimetric inequalities and they obtain the following inequality for a function f such that $f \in BV(\mathbb{R}^n)$ and $f \in \dot{B}^{-\beta,\infty}_{\infty}(\mathbb{R}^n)$

$$\|f\|_{\dot{W}^{s_1,q}} \leq C\|f\|^{1/q}_{BV}\|f\|^{1-1/q}_{\dot{B}^{-\beta,\infty}_{\infty}}, \tag{3}$$

where $1 < q \leq 2$, $0 \leq s_1 < 1/q$ and $\beta = (1 - s_1 q)/(q-1)$. When $s_1 = 0$, this last result implies (2) with $p = 1$, but is limited by the fact that $1 < q \leq 2$. Indeed, in order to obtain (3), it is necessary to consider a Besov space of type $\dot{B}^{s_1,q}_2$ in the left-hand side of the inequality (3) and since we have $\dot{B}^{s_1,q}_{\min\{2,q\}} \subset \dot{W}^{s_1,q} \subset \dot{B}^{s_1,q}_{\max\{2,q\}}$, by this method it is not possible to treat the case when $q > 2$. Furthermore, the geometric arguments such as isoperimetric inequalities used in [16] considerably reduce the possibility to generalize this inequality to other settings. In particular, for the general framework of stratified Lie groups that will be used here (which are very natural generalizations of the Euclidean space $\mathbb{R}^n$), it is not possible to apply those arguments. Concerning some geometric issues of stratified Lie groups see [20] or [37].

Finally, there is the following weak-type inequality given in [11].

$$\|f\|_{\dot{W}^{s,q}_{\infty}} \leq C\|\nabla f\|^{\theta}_{L^1}\|f\|^{1-\theta}_{\dot{B}^{-\beta,\infty}_{\infty}}, \tag{4}$$

where $1 < q < +\infty$, $0 < s < 1/q < 1$, $\theta = 1/q$ and $\beta = \frac{1-sq}{q-1}$. The proof of this inequality relies in spectral theory and it was given in the setting of stratified Lie groups, see also [12] for more general frameworks. However, even though we can consider the Sobolev space $\dot{W}^{1,1}$ in the right-hand side of (4) and we have a Sobolev space in the left-hand side of the inequality and we do not have any restriction on the parameter q (here $1 < q < +\infty$), the inequality (4) is of *weak* type (in the sense that the Sobolev space $\dot{W}^{s,q}_{\infty}$ is based on the weak L^q space) and the proof given in [11] does not allow us to treat *strong* inequalities with usual Sobolev spaces.

To the best of our knowledge, a strong version of inequality (4) and a generalization of inequality (3) to the case $q > 2$ is an open point and in this article we will study these improved Sobolev inequalities in the case when $p = 1$. Our first result is then a generalization of the inequalities (2), (3) and (4), and this generalization is twofold: first, it is possible to consider global parameters (*i.e.* $1 < q < +\infty$) for (strong) Sobolev spaces $\dot{W}^{s,q}$ in the left-hand side of inequalities and second, the proof exposed here can be easily applied to more general frameworks than the Euclidean one as the tools used rely only on harmonic analysis arguments.

Here is our first theorem, which is stated for simplicity in $\mathbb{R}^n$, however the proof will be carried out in the setting of stratified Lie groups.

Theorem 1 *Let $f : \mathbb{R}^n \longrightarrow \mathbb{R}$ be a function such that $\nabla f \in L^1(\mathbb{R}^n)$ and $f \in \dot{B}^{-\beta,\infty}_{\infty}(\mathbb{R}^n)$. Then we have the following inequality:*

$$\|f\|_{\dot{W}^{s,q}} \leq C\|\nabla f\|^{\theta}_{L^1}\|f\|^{1-\theta}_{\dot{B}^{-\beta,\infty}_{\infty}}, \tag{5}$$

where $1 < q < +\infty$, $0 \leq s < 1/q$, $\beta = \frac{1-sq}{q-1}$ and $\theta = \frac{1}{q}$.

This inequality implies (2) when $s = 0$ and (3) when $1 < q \leq 2$. Moreover, since we have the space inclusion $L^p \subset L^{p,\infty}$, we obtain immediately (4). In Section 7 we will also study a A_1-weighted version of the previous result with the Theorem 6.

Let us mention now that in the case of Lorentz spaces $L^{p,q}(\mathbb{R}^n)$, H. Bahouri and A. Cohen [4] proved the inequality

$$\|f\|_{L^{p,q}} \leq C\|f\|_{\dot{B}^{s,q}_q}^{q/p}\|f\|_{\dot{B}^{s-n/q,\infty}_q}^{1-q/p} \qquad \text{with } \frac{1}{p} = \frac{1}{q} - \frac{s}{n}. \tag{6}$$

Remark that in this estimate, the index q defining the Lorentz and the Besov spaces is related to the parameters p, s and the dimension n. This inequality was generalized by D. Chamorro & P-G. Lemarié-Rieusset [13] for other values of the parameter q using interpolation techniques and pointwise estimates. In a recent article, V.I. Kolyada and F.J. Pérez Lázaro [24] gave an interesting proof for inequalities of type (1), (2) and (6) based on the use of rearrangement inequalities and the properties of the Gauss-Weierstrass kernel. However, inequalities with Sobolev spaces in the left-hand side in the spirit of (3) or (4) seemed to be out of the range of this work.

Motivated by the use of Lorentz spaces in these previous works, in our second theorem we will provide a generalization of improved Sobolev inequalities of type (1) by considering weighted Lorentz-based Sobolev spaces defined as the set of measurable functions $f : \mathbb{R}^n \longrightarrow \mathbb{R}$ such that the quantity $\|(-\Delta)^{\frac{s}{2}}f\|_{\Lambda^p(w)}$ is bounded where for $s > 0$ the fractional power of the Laplacian is defined in the Fourier level by $\widehat{(-\Delta)^{\frac{s}{2}}f}(\xi) = c|\xi|^s\widehat{f}(\xi)$, and for $1 < p < +\infty$ the space $\Lambda^p(w)$ denotes the classical Lorentz space of functions introduced in [26] and [27] defined as

$$\Lambda^p(w) = \left\{ f : \|f\|_{\Lambda^p(w)} = \left(\int_0^{+\infty} f^*(t)^p w(t)dt\right)^{1/p} < +\infty \right\},$$

where w is a weight in $\mathbb{R}_+$ and f^* denotes the nonincreasing rearrangement of f (see [5] for standard notations). Many of the properties of these spaces depend on the weight w: in particular, if $w = 1$ we have $\Lambda^p(w) = L^p$ and if $w(t) = t^{p/q-1}$, with $1 \leq q \leq +\infty$, we obtain $\Lambda^p(w) = L^{q,p}$, where $L^{q,p}$ are the usual Lorentz spaces. In this work we will consider the weighted Lorentz space $\Lambda^p(w)$ such that the weight w satisfies the B_p condition, the reason for this is given by the fact that M. A. Ariño and B. Muckenhoupt showed in [3] that this B_p condition characterizes the boundedness of the Hardy-Littlewood maximal operator on $\Lambda^p(w)$ and this particular property will be intensively used in our proofs. See Section 4 for definitions and [3], [8], [10] and [34] for more details and properties concerning these functional spaces.

In this direction of generalization, standard Sobolev inequalities have been studied by A. Cianchi [15] in the context of Orlicz-Sobolev spaces, but we think that inequalities of the general type (7) presented in Theorem 2 below are new. Again, for the sake of simplicity, we state our result in the euclidean setting of $\mathbb{R}^n$ but the proof will be given in the framework of stratified Lie groups.

Theorem 2 *Let $s > 0$, $w \in B_p$ be a weight and let $f : \mathbb{R}^n \longrightarrow \mathbb{R}$ be a function such that $(-\Delta)^{\frac{s}{2}}f \in \Lambda^p(w)(\mathbb{R}^n)$ and $f \in \dot{B}^{-\beta,\infty}_{\infty}(\mathbb{R}^n)$. Then we have the following version of improved Sobolev inequalities:*

$$\|(-\Delta)^{\frac{s_1}{2}}f\|_{\Lambda^q(w)} \leq C\|(-\Delta)^{\frac{s}{2}}f\|_{\Lambda^p(w)}^{\theta}\|f\|_{\dot{B}^{-\beta,\infty}_{\infty}}^{1-\theta}, \tag{7}$$

where $1 < p < q < +\infty$, $\theta = p/q$, $s_1 = \theta s - (1-\theta)\beta$ and $-\beta < s_1 < s$.

The choice of the weights in the B_p class is given by two important facts. First, these weights allow us to consider general functional spaces, in particular we can easily recover standard Lorentz spaces. Second, these weights ensure that maximal function is bounded in the spaces $\Lambda^p(w)$ for $1 < p < +\infty$, and this feature is crucial as the proof of Theorem 2 requires this property. Note in particular that inequality (7) is different from inequality (6) since Lorentz-Sobolev spaces are not included in the scale of Besov spaces.

Finally, since our proof of Theorem 2 relies essentially on a pointwise inequality and on the boundedness of the Hardy-Littlewood maximal operator, it is possible to give a related result replacing classical Lorentz spaces by Morrey spaces $\mathcal{M}^{p,a}(\mathbb{R}^n)$ which are a useful generalization of Lebesgue spaces. Classical Hardy-Littlewood-Sobolev inequalities were studied in this functional framework by D. Adams [1] and by F. Chiarenza & M. Frasca [14] and our next theorem is an improvement of these inequalities.

Theorem 3 *Let $s > 0$, $1 < p < +\infty$ and $0 \leq a < n$ and let f be a function such that $(-\Delta)^{\frac{s}{2}} f \in \mathcal{M}^{p,a}(\mathbb{R}^n)$ and $f \in \dot{B}^{-\beta,\infty}_{\infty}(\mathbb{R}^n)$. Then we have*

$$\|(-\Delta)^{\frac{s_1}{2}} f\|_{\mathcal{M}^{q,a}} \leq C \|(-\Delta)^{\frac{s}{2}} f\|^{\theta}_{\mathcal{M}^{p,a}} \|f\|^{1-\theta}_{\dot{B}^{-\beta,\infty}_{\infty}}, \tag{8}$$

where $1 < p < q < +\infty$, $\theta = p/q$, $s_1 = \theta s - (1-\theta)\beta$ and $-\beta < s_1 < s$.

The plan of this article is the following. In Section 2 we present our general framework which is given by stratified Lie groups. These groups are quite natural generalization of $\mathbb{R}^n$ but they present some particularities that should be taken into account in the computations. Section 3 is devoted to one of our most important tool: the spectral decomposition of the Laplacian that will allow us to build new operators in a very simple way and whose associated kernels will enjoy of useful properties. In Section 4 we give the precise definition of all the functional spaces used in the previous inequalities and in Section 5 - 6 we present the proofs of Theorems 1 and Theorem 2 respectively. Finally, in Section 7 we give the proof of Theorem 3 and some variations of the previous results.

2 Stratified Lie groups: notation and basic properties

As said in the introduction, stratified Lie groups are natural generalizations of $\mathbb{R}^n$ when considering general dilation structures. Although stratified Lie groups share common features with $\mathbb{R}^n$, there are some special points that must be taken into account: for example these groups are no longer abelian and this fact requires to be carefull in some computations, furthermore from the geometric point of view, the inner geometric structure of these groups can be very different from the euclidean setting. It is then necessary to recall some basic facts about stratified Lie groups, for further information see [17], [18], [39], [36] and the references given there in.

We start with the notion of *homogeneous group* $\mathbb{G}$ which is the data of $\mathbb{R}^n$ equipped with a structure of Lie group and we will always suppose that the origin is the identity. We define a *dilation structure* by fixing integers $(a_i)_{1\leq i\leq n}$ such that $1 = a_1 \leq ... \leq a_n$ and by writing:

$$\begin{array}{rcl} \delta_\alpha : \mathbb{R}^n & \longrightarrow & \mathbb{R}^n \\ x & \longmapsto & \delta_\alpha[x] = (\alpha^{a_1} x_1, ..., \alpha^{a_n} x_n). \end{array} \tag{9}$$

We will often note αx instead of $\delta_\alpha[x]$ and α will always indicate a strictly positive real number.

Of course, the Euclidean space $\mathbb{R}^n$ with its group structure and provided with its usual dilations (i.e. $a_i = 1$, for $i = 1, ..., n$) is a homogeneous group. Here is another example: if $x = (x_1, x_2, x_3)$ is an element of $\mathbb{R}^3$, we can fix a dilation by writing $\delta_\alpha[x] = (\alpha x_1, \alpha x_2, \alpha^2 x_3)$ for $\alpha > 0$. Then, the well suited group law with respect to this dilation is given by $x \cdot y = (x_1, x_2, x_3) \cdot (y_1, y_2, y_3) = (x_1 + y_1, x_2 + y_2, x_3 + y_3 + \frac{1}{2}(x_1 y_2 - y_1 x_2))$. Remark in particular that this group law is no longer abelian. The triplet $(\mathbb{R}^3, \cdot, \delta)$ corresponds to the Heisenberg group $\mathbb{H}^1$ which is the first non-trivial example of a homogeneous group.

The *homogeneous dimension* with respect to dilation structure (9) is given by $N = \sum\limits_{1\leq i\leq n} a_i$. We observe that it is always larger than the topological dimension n since each integer a_i verifies $a_i \geq 1$ for all $i = 1, ..., n$. For instance, in the Heisenberg group $\mathbb{H}^1$ we have $N = 4$ and $n = 3$ while in the Euclidean case these two concepts coincide. Now we will say that a function on $\mathbb{G} \setminus \{0\}$ is *homogeneous* of degree $\lambda \in \mathbb{R}$ if $f(\delta_\alpha[x]) = \alpha^\lambda f(x)$ for all $\alpha > 0$. In the same way, we will say that a differential operator D is homogeneous of degree λ if

$$D(f(\delta_\alpha[x])) = \alpha^\lambda (Df)(\delta_\alpha[x]),$$

for all f in operator's domain. In particular, if f is homogeneous of degree λ and if D is a differential operator of degree μ, then Df is homogeneous of degree $\lambda - \mu$.

The presence of a dilation structure is one of most important features of stratified Lie groups and the homogeneity with respect to these dilations will play a useful role in our computations.

From the point of view of measure theory, homogeneous groups behave in a traditional way since Lebesgue measure dx is bi-invariant and coincides with the Haar measure, thus for any subset E of $\mathbb{G}$ we will note its measure as $|E|$. This fact also allows us to define Lebesgue spaces in a classical way (see also Section 4 below).

The convolution will be a very useful tool in our computations, and for two functions f and g on $\mathbb{G}$ it is defined by

$$f * g(x) = \int_{\mathbb{G}} f(y)g(y^{-1} \cdot x)dy = \int_{\mathbb{G}} f(x \cdot y^{-1})g(y)dy, \quad x \in \mathbb{G}.$$

However, since the group law of a stratified Lie group is not necessarily commutative, we do not have in general the identity $f * g = g * f$ and we need to take care of this fact. Nevertheless, we have at our disposal Young's inequalities:

Lemma 2.1 *If $1 \leq p, q, r \leq +\infty$ such that $1 + \frac{1}{q} = \frac{1}{p} + \frac{1}{r}$. If $f \in L^p(\mathbb{G})$ and $g \in L^r(\mathbb{G})$, then $f * g \in L^q(\mathbb{G})$ and*

$$\|f * g\|_{L^q} \leq \|f\|_{L^p} \|g\|_{L^r}.$$

A proof is given in [18]. A weak version of Young's inequalities will be stated in Section 4.

For a homogeneous group $\mathbb{G} = (\mathbb{R}^n, \cdot, \delta)$ we consider now its Lie algebra $\mathfrak{g}$ whose elements can be conceived in two different ways: as *left*-invariant vector fields or as *right*-invariant vector fields. The left-invariant vectors fields $(X_j)_{1 \leq j \leq n}$ are determined by the formula

$$(X_j f)(x) = \left.\frac{\partial f(x \cdot y)}{\partial y_j}\right|_{y=0} = \frac{\partial f}{\partial x_j} + \sum_{j<k} q_j^k(x) \frac{\partial f}{\partial x_k},$$

where $q_j^k(x)$ is a homogeneous polynomial of degree $a_k - a_j$ and f is a smooth function on $\mathbb{G}$. By this formula one deduces easily that these vectors fields are homogeneous of degree a_j and we have $X_j\left(f(\alpha x)\right) = \alpha^{a_j}(X_j f)(\alpha x)$. We will note $(Y_j)_{1 \leq j \leq n}$ the right invariant vector fields defined in a totally similar way:

$$(Y_j f)(x) = \left.\frac{\partial f(y \cdot x)}{\partial y_j}\right|_{y=0}.$$

A homogeneous group $\mathbb{G}$ is *stratified* if its Lie algebra $\mathfrak{g}$ breaks up into a sum of linear subspaces $\mathfrak{g} = \bigoplus_{1 \leq j \leq k} E_j$ such that E_1 generates the algebra $\mathfrak{g}$ and $[E_1, E_j] = E_{j+1}$ for $1 \leq j < k$ and $[E_1, E_k] = \{0\}$ and $E_k \neq \{0\}$, but $E_j = \{0\}$ if $j > k$. Here $[E_1, E_j]$ indicates the subspace of $\mathfrak{g}$ generated by the elements $[U, V] = UV - VU$ with $U \in E_1$ and $V \in E_j$. The integer k is called the *degree* of stratification of $\mathfrak{g}$. For example, on Heisenberg group $\mathbb{H}^1$, we have $k = 2$ while in the Euclidean case $k = 1$.

We will suppose from now on that $\mathbb{G}$ is **stratified**. Within this framework, we will fix once and for all the family of vectors fields

$$\mathbf{X} = \{X_1, ..., X_m\},$$

such that $a_1 = a_2 = \ldots = a_m = 1$ $(m < n)$, then the family $\mathbf{X}$ is a base of E_1 and generates the Lie algebra of $\mathfrak{g}$, which is precisely the Hörmander's condition (see [18] and [39]) and this particular choice ensures several important properties, in particular to the family $\mathbf{X}$ is associated the Carnot-Carathéodory distance d which is left-invariant and compatible with the topology on $\mathbb{G}$ (see [39] for more details) and for any $x \in \mathbb{G}$ we will denote by $|x| = d(x, e)$ and for $r > 0$ we form open balls by writing $B(x, r) = \{y \in \mathbb{G} : d(x, y) < r\}$. By simple homogeneity arguments we obtain that stratified Lie groups have polynomial volume growth since we have $|B(\cdot, r)| = r^N |B(\cdot, 1)|$.

The main tools of this paper depend on the properties of the gradient, the Laplacian and the associated heat kernel, but before introducing them, we make here some remarks on general vectors fields X_j and Y_j.

Let us fix some notation: for any multi-index $I = (i_1, ..., i_n) \in \mathbb{N}^n$, one defines X^I by $X^I = X_1^{i_1} \ldots X_n^{i_n}$ and Y^I by $Y^I = Y_1^{i_1} \ldots Y_n^{i_n}$, furthermore we denote by $|I| = i_1 + \ldots + i_n$ the order of the derivation of the operators X^I or Y^I and $d(I) = a_1 i_1 + \ldots + a_n i_n$ the homogeneous degree of these ones. Now, for $\varphi, \psi \in \mathcal{C}_0^\infty(\mathbb{G})$ we have the equality

$$\int_{\mathbb{G}} \varphi(x)(X^I \psi)(x)dx = (-1)^{|I|} \int_{\mathbb{G}} (X^I \varphi)(x)\psi(x)dx.$$

The interaction of operators X^I and Y^I with convolutions is clarified by the following identities:

$$X^I(f * g) = f * (X^I g), \qquad Y^I(f * g) = (Y^I f) * g, \qquad (X^I f) * g = f * (Y^I g). \tag{10}$$

Finally, one will say that a function $f \in \mathcal{C}^\infty(\mathbb{G})$ belongs to the Schwartz class $\mathcal{S}(\mathbb{G})$ if the following semi-norms are bounded for all $k \in \mathbb{N}$ and any multi-index I: $N_{k,I}(f) = \sup_{x\in\mathbb{G}} (1 + |x|)^k |X^I f(x)|$.

Remark 2.1 To characterize the Schwartz class $\mathcal{S}(\mathbb{G})$ we can replace vector fields X^I in the semi-norms $N_{k,I}$ above by right-invariant vector fields Y^I.

For a proof of these facts and for further details see [18] and [19].

We define now the *gradient* on $\mathbb{G}$ from vectors fields of homogeneity degree equal to one (*i.e.* those composing the family $\mathbf{X}$) by fixing

$$\nabla = (X_1, ..., X_m).$$

This operator is of course left invariant and homogeneous of degree 1. The length of the gradient is given by the formula $|\nabla f| = \big((X_1 f)^2 + ... + (X_m f)^2\big)^{1/2}$. We also define the right invariant gradient $\widetilde{\nabla} = (Y_1, ..., Y_m)$, and using (10) we have the identity

$$(\nabla f) * g = f * (\widetilde{\nabla} g).$$

We define now the Laplacian we are going to work with. Let us notice that in this setting there is not a single way to build a Laplacian, see for example [19]. In this article we will use the Laplacian, denoted by $\mathcal{J}$, which is given from the family $\mathbf{X}$ in the following way

$$\mathcal{J} = \nabla^* \nabla = -\sum_{j=1}^{m} X_j^2. \tag{11}$$

This is a positive self-adjoint, hypo-elliptic operator (since the family $\mathbf{X}$ satisfies the Hörmander's condition), having as domain of definition $L^2(\mathbb{G})$. Its associated *heat operator* on $\mathbb{G}\times]0, +\infty[$ is given by $\partial_t + \mathcal{J}$.

We recall now some well-known properties of the heat operator and its associated kernel.

Theorem 4 *There exists a unique family of continuous linear operators $(H_t)_{t>0}$ defined on $L^1 + L^\infty(\mathbb{G})$ with the semi-group property $H_{t+s} = H_t H_s$ for all $t, s > 0$ and $H_0 = Id$, such that:*

1) *the Laplacian $\mathcal{J}$ is the infinitesimal generator of the semi-group $H_t = e^{-t\mathcal{J}}$;*
2) *H_t is a contraction operator on $L^p(\mathbb{G})$ for $1 \leq p \leq +\infty$ and for $t > 0$;*
3) *the semi-group H_t admits a convolution kernel $H_t f = f * h_t$ where $h_t(x) = h(x, t) \in \mathcal{C}^\infty(\mathbb{G}\times]0, +\infty[)$ is the heat kernel which satisfies the following points:*
   (a) *$(\partial_t + \mathcal{J})h_t = 0$ on $\mathbb{G}\times]0, +\infty[$, and $h(x, t) = h(x^{-1}, t)$, $h(x, t) \geq 0$ and $\int_{\mathbb{G}} h(x, t)dx = 1$,*
   (b) *h_t has the semi-group property: $h_t * h_s = h_{t+s}$ for $t, s > 0$ and we have $h(\delta_\alpha[x], \alpha^2 t) = \alpha^{-N} h(x, t)$,*
   (c) *For every $t > 0$, $x \mapsto h(x, t)$ belong to the Schwartz class in $\mathbb{G}$.*
4) *$\|H_t f - f\|_{L^p} \to 0$ if $t \to 0$ for $f \in L^p(\mathbb{G})$ and $1 \leq p < +\infty$;*
5) *For $\varphi \in \mathcal{C}^\infty(\mathbb{G})$ and for $t > 0$ we have $\mathcal{J} H_t(f) = H_t \mathcal{J}(f)$.*

For a detailed proof of these and other important facts concerning the heat semi-group see [18] and [32].

To close this section we recall the definition of the Laplacian's fractional powers. If $s > 0$ we write

$$\mathcal{J}^s f(x) = \frac{1}{\Gamma(k - s)} \int_0^{+\infty} t^{k-s-1} \mathcal{J}^k H_t f(x) dt, \tag{12}$$

for all $f \in \mathcal{C}^\infty(\mathbb{G})$ with k an integer greater than s. This formula will be used in the sequel, but as we will see in the next section, there is another way to define the fractional powers of the Laplacian.

3 Spectral tools

Since the Laplacian given by (11) is a positive self-adjoint operator, it admits a spectral decomposition of the following form

$$\mathcal{J} = \int_0^{+\infty} \lambda dE_\lambda.$$

This spectral decomposition allows us to define the fractional powers of the Laplacian by the expression

$$\mathcal{J}^s = \int_0^{+\infty} \lambda^s dE_\lambda, \text{ with } s > 0.$$

This formula is very useful to deduce some properties for the fractional Laplacian like $\mathcal{J}^{s_1}[\mathcal{J}^{s_2} f] = \mathcal{J}^{s_1+s_2} f$ for $s_1, s_2 > 0$ whenever these quantities are well defined. But it will also help us to build a family of operators $m(\mathcal{J})$ associated to a Borel function m. A classical example of this situation is given by the heat semi-group

$$H_t = e^{-t\mathcal{J}} = \int_0^{+\infty} e^{-t\lambda} dE_\lambda \qquad \text{with } m(\lambda) = e^{-\lambda},$$

and from these formulas we can see that we have the identity $\mathcal{J}^s H_t = H_t \mathcal{J}^s$. It is however possible to go one step further, indeed, following [23] and [19] we have the next result which is the key of many of our computations.

Proposition 3.1 ([19], Proposition 6) *Let $k \in \mathbb{N}$ and m be a function of class $\mathcal{C}^k(\mathbb{R}^+)$, we write*

$$\|m\|_{(k)} = \sup_{\substack{0 \leq r \leq k \\ \lambda > 0}} (1+\lambda)^k |m^{(r)}(\lambda)|.$$

We define the operator $m(t\mathcal{J})$ for $t > 0$ by the expression $m(t\mathcal{J}) = \int_0^{+\infty} m(t\lambda) dE_\lambda$. Then this operator admits a convolution kernel $M_t(x) = t^{-N/2} M(t^{-1/2} x)$ and moreover, for $\alpha \in \mathbb{R}$ and I a multi-index, there exists $C > 0$ and $k \in \mathbb{N}$ such that:

$$\big\|(1 + \| \cdot \|)^\alpha X^I M_t(\cdot)\big\|_{L^p} \leq C\, t^{-\frac{N}{2p'} - \frac{|I|}{2}} (1 + \sqrt{t})^\alpha \|m\|_{(k)} \qquad \text{with } \frac{1}{p} + \frac{1}{p'} = 1.$$

By homogeneity, the conclusion of this proposition remains true if instead of left-invariant vector fields X^I we consider right-invariant vector fields Y^I.

Corollary 3.1 *Let m be the restriction on $\mathbb{R}^+$ of a function in $\mathcal{S}(\mathbb{R})$. Then the kernel M of the operator $m(\mathcal{J})$ is in $\mathcal{S}(\mathbb{G})$.*

4 Functional spaces

We give in this section the precise definition of all the functional spaces involved in Theorems 1 and 2. In a general way, given a norm $\| \cdot \|_X$, we will define the corresponding functional space $X(\mathbb{G})$ by $\{f \in \mathcal{S}'(\mathbb{G}) : \|f\|_X < +\infty\}$. The constant that appear in this paper such as C may change from one occurrence to the next.

- **Lebesgue spaces** $L^p(\mathbb{G})$. For a measurable function $f : \mathbb{G} \longrightarrow \mathbb{R}$ and for $1 \leq p < +\infty$ we define Lebesgue space by the norm $\|f\|_{L^p} = \left(\int_{\mathbb{G}} |f(x)|^p dx\right)^{1/p}$, while for $p = +\infty$ we have $\|f\|_{L^\infty} = \sup_{x \in \mathbb{G}} ess |f(x)|$. Let us notice that we also have the following characterization using the distribution function:

$$\|f\|_{L^p}^p = p \int_0^{+\infty} \alpha^{p-1} |\{x \in \mathbb{G} : |f(x)| > \alpha\}| d\alpha.$$

- **weak-Lebesgue spaces** $L^{p,\infty}(\mathbb{G})$. We define them as the set of all measurable functions $f:\mathbb{G}\longrightarrow\mathbb{R}$ such that

$$\|f\|_{L^{p,\infty}}=\sup_{\alpha>0}\{\alpha:|\{x\in\mathbb{G}:|f(x)|>\alpha\}|^{1/p}\}$$

is finite.

We will need the following version of Young's inequality where weak L^p spaces are involved:

Lemma 4.1 *Let $p,q,r>1$. If $f\in L^{p,\infty}(\mathbb{G})$ and if $g\in L^r(\mathbb{G})$, then $f*g\in L^q(\mathbb{G})$ with $1+\frac{1}{q}=\frac{1}{p}+\frac{1}{r}$ and we have the inequality*

$$\|f*g\|_{L^q}\leq\|f\|_{L^{p,\infty}}\|g\|_{L^r}.$$

See a proof of this Lemma in [22], Theorem 1.4.24.

- **Sobolev spaces** $\dot{W}^{s,p}(\mathbb{G})$. If $1<p<+\infty$ and for $s>0$ we have

$$\|f\|_{\dot{W}^{s,p}}=\|\mathcal{J}^{\frac{s}{2}}f\|_{L^p},$$

while if $p=s=1$ we will note

$$\|f\|_{\dot{W}^{1,1}}=\|\nabla f\|_{L^1}.$$

- **weak Sobolev spaces** $\dot{W}^{s,p}_{\infty}(\mathbb{G})$. These spaces are defined just as classical Sobolev spaces, but we replace the L^p norm by the weak L^p one in the following manner:

$$\|f\|_{\dot{W}^{s,p}_{\infty}}=\|\mathcal{J}^{s/2}f\|_{L^{p,\infty}}\quad\text{with } 1<p<+\infty \text{ and } s>0.$$

- **Besov spaces** $\dot{B}^{s,q}_{p}(\mathbb{G})$. There are many different (and equivalent) ways to define these spaces in the setting of stratified Lie groups. In this article we will mainly use the thermic definition given by

$$\|f\|_{\dot{B}^{s,q}_{p}}=\left(\int_0^{+\infty}t^{(m-s/2)q}\left\|\frac{\partial^m H_tf}{\partial t^m}(\cdot)\right\|^q_{L^p}\frac{dt}{t}\right)^{1/q},$$

for $1\leq p,q\leq+\infty, s>0$ and m an integer such that $m>s/2$. For Besov spaces of indices $(-\beta,\infty,\infty)$ which appear in all the improved Sobolev inequalities we have:

$$\|f\|_{\dot{B}^{-\beta,\infty}_{\infty}}=\sup_{t>0}\ t^{\beta/2}\|H_tf\|_{L^\infty}.$$

- **Lorentz spaces** $\Lambda^p(w)(\mathbb{G})$. Let $f:\mathbb{G}\longrightarrow\mathbb{R}$ be a measurable function. We define f^*, the nonincreasing rearrangement of the function f, by the expression $f^*(t)=\inf\{\alpha\geq 0:|\{x\in\mathbb{G}:|f(x)|>\alpha\}|\leq t\}$. We will say that a nonnegative locally integrable function $w:\mathbb{R}^+\longrightarrow\mathbb{R}^+$ belongs to the Ariño-Muckenhoupt class B_p for $1\leq p<+\infty$, if there exists $C>0$ such that

$$\int_r^{+\infty}\left(\frac{r}{t}\right)^p w(t)dt\leq C\int_0^r w(t)dt,\ \text{for all } 0<r<+\infty.$$

It is not difficult to see that if $0<p<q<+\infty$, then we have the inclusion of classes $B_p\subset B_q$. We define the Lorentz spaces $\Lambda^p(w)$ with $1\leq p<+\infty$ by the formula

$$\|f\|_{\Lambda^p(w)}=\left(\int_0^{+\infty}(f^*(t))^p w(t)dt\right)^{\frac{1}{p}}.$$

As said in the introduction, the choice of the B_p class is due to the fact that the Hardy-Littlewood maximal operator $\mathcal{M}_B$, given for a measurable function f by

$$\mathcal{M}_Bf(x)=\sup_{B\ni x}\frac{1}{|B|}\int_B|f(y)|dy,\quad\text{where } B \text{ is an open ball,}\tag{13}$$

is bounded on the spaces $\Lambda^p(w)$ for $1<p<+\infty$: $\|\mathcal{M}_Bf\|_{\Lambda^p(w)}\leq C\|f\|_{\Lambda^p(w)}$, where C is depending on the quantity

$$[w]_{B_p}=\sup_{r>0}\left\{r^p\left(\int_r^{+\infty}\frac{w(t)}{t^p}dt\right)/\left(\int_0^r w(t)dt\right)\right\}.$$

For more properties of these weights and the associated classical Lorentz spaces see [3], [8], [34] and [10].

- **Lorentz-Sobolev spaces** $\dot{\Lambda}^{s,p}(w)(\mathbb{G})$. Once we have fixed the base space $\Lambda^p(w)$, the homogeneous Lorentz-Sobolev spaces are easy to define and are given for $1 < p < +\infty$ and for $s > 0$ in the following way

$$\|f\|_{\dot{\Lambda}^{s,p}(w)} = \left(\int_0^{+\infty} \left((\mathcal{J}^{\frac{s}{2}} f)^*(t)\right)^p w(t)dt\right)^{\frac{1}{p}}.$$

- **weak Lorentz spaces** $\Lambda^{p,\infty}(w)(\mathbb{G})$. Let w a weight in $\mathbb{R}^+$. For $0 < p < +\infty$, the weak Lorentz space $\Lambda^{p,\infty}(w)$ is the class of all measurable functions $f : \mathbb{G} \longrightarrow \mathbb{R}$ such that

$$\|f\|_{\Lambda^{p,\infty}(w)} = \sup_{t>0} f^*(t) W^{1/p}(t) < +\infty,$$

where $W(t) = \displaystyle\int_0^t w(s)ds$. The weak Lorentz spaces were introduced in [8] and further investigated in [7], [6] and [9]. The problem of characterizing when the weak type Lorentz spaces $\Lambda^{p,\infty}(w)$, $0 < p < +\infty$ are Banach spaces was studied in [34].

- **weak Lorentz-Sobolev spaces** $\dot{\Lambda}^{s,p,\infty}(w)(\mathbb{G})$. For $1 < p < +\infty$ the homogeneous weak Lorentz-Sobolev spaces are given by

$$\|f\|_{\dot{\Lambda}^{s,p,\infty}(w)} = \sup_{t>0} (\mathcal{J}^{\frac{s}{2}} f)^*(t) \left(\int_0^t w(s)ds\right)^{1/p}.$$

- **Morrey spaces** $\mathcal{M}^{p,a}(\mathbb{G})$. For $1 < p < +\infty$ and $0 \leq a < N$, we define Morrey spaces as the space of locally integrable functions such that

$$\|f\|_{\mathcal{M}^{p,a}} = \sup_{x_0 \in \mathbb{R}^n} \sup_{0<r<+\infty} \left(\frac{1}{r^a} \int_{B(x_0,r)} |f(x)|^p dx\right)^{\frac{1}{p}} < +\infty.$$

Morrey spaces are indeed a generalization of Lebesgue spaces since when $a = 0$ we have $\mathcal{M}^{p,0} \simeq L^p$. The use of Morrey and Morrey-Sobolev spaces in this article is due to the fact that the Hardy-Littlewood maximal operator (13) is also bounded in such spaces. See more details in [14] in the framework of $\mathbb{R}^n$ and [2] in the setting of stratified Lie groups. See also [33] and the references given there in for other interesting generalizations.

- **Morrey-Sobolev spaces** $\dot{\mathcal{M}}^{s,p,a}(\mathbb{G})$. For $0 < s$ and $1 < p < +\infty$ with $0 \leq a < N$ we consider the homogeneous Morrey-Sobolev spaces $\dot{\mathcal{M}}^{s,p,a}$ by the quantity

$$\|f\|_{\dot{\mathcal{M}}^{s,p,a}} = \|\mathcal{J}^{\frac{s}{2}} f\|_{\mathcal{M}^{p,a}}.$$

5 Proof of Theorem 1

Theorem 1 will be a consequence of the following result which is stated in the general setting of stratified Lie groups.

Theorem 5 *Let $f : \mathbb{G} \longrightarrow \mathbb{R}$ be a function such that $\nabla f \in L^1(\mathbb{G})$ and $f \in \dot{B}^{-\beta,\infty}_{\infty}(\mathbb{G})$. Then we have the following inequality:*

$$\|f\|_{\dot{W}^{s,q}} \leq C\|\nabla f\|^{\theta}_{L^1} \|f\|^{1-\theta}_{\dot{B}^{-\beta,\infty}_{\infty}}, \tag{14}$$

where $1 < q < +\infty$, $0 \leq s < 1/q$, $\beta = \frac{1-sq}{q-1}$ and $\theta = \frac{1}{q}$.

We assume for the time being that the quantity $\|\mathcal{J}^{\frac{s}{2}} f\|_{L^q}$ is finite, we will see how to discard this extra hypothesis with the Proposition 5.3.

We start our proof by remarking that the operator $\mathcal{J}^{\frac{s}{2}}$ carries out an isomorphism between the Besov spaces $\dot{B}^{-\beta,\infty}_{\infty}(\mathbb{G})$ and $\dot{B}^{-\beta-s,\infty}_{\infty}(\mathbb{G})$ (see [32]) and thus we can rewrite inequality (14) in the following manner

$$\|f\|_{\dot{W}^{s,q}} \leq C\|\nabla f\|^{\theta}_{L^1} \|\mathcal{J}^{\frac{s}{2}} f\|^{1-\theta}_{\dot{B}^{-\beta-s,\infty}_{\infty}}.$$

By homogeneity of this inequality we can assume with no loss of generality that

$$\|\mathcal{J}^{\frac{s}{2}} f\|_{\dot{B}^{-\beta-s,\infty}_{\infty}} \leq 1, \tag{15}$$

and with such an assumption we only have to prove the inequality

$$\|f\|_{\dot{W}^{s,q}} \leq C\|\nabla f\|^{\theta}_{L^1},$$

where $C = C(q)$ is a positive constant independent from the function f.

Now, if we define $t > 0$ by

$$t_\alpha = \alpha^{-\frac{2}{\beta+s}} \text{ with } \alpha > 0, \tag{16}$$

using the thermic definition of the Besov space $\dot{B}^{-\beta-s,\infty}_{\infty}$ we have

$$\|\mathcal{J}^{\frac{s}{2}} f\|^{1-\theta}_{\dot{B}^{-\beta-s,\infty}_{\infty}} \leq 1 \iff \sup_{t_\alpha>0} t_\alpha^{\frac{\beta+s}{2}} \|H_{t_\alpha}\mathcal{J}^{\frac{s}{2}} f\|_{L^\infty} \leq 1 \iff \|H_{t_\alpha}\mathcal{J}^{\frac{s}{2}} f\|_{L^\infty} \leq t_\alpha^{-\frac{\beta+s}{2}} \iff \|H_{t_\alpha}\mathcal{J}^{\frac{s}{2}} f\|_{L^\infty} \leq \alpha. \tag{17}$$

To continue, we consider a smooth non-negative function $\theta_0 : \mathbb{R}^+ \longrightarrow \mathbb{R}^+$ such that

$$\theta_0(\lambda) = \begin{cases} 1 \text{ if } \lambda \in [0, 1/2[, \\ 0 \text{ if } \lambda \in [1, +\infty[, \end{cases} \qquad \text{and we define } \theta_1(\lambda) = 1 - \theta_0(\lambda). \tag{18}$$

Using the spectral resolution of the Laplacian, we can associate to these functions θ_0 and θ_1 two convolution operators $\theta_0(\mathcal{J})$ and $\theta_1(\mathcal{J})$ with kernels Θ_0 and Θ_1 in the following manner:

$$\theta_0(\mathcal{J})(\phi) = \left(\int_0^{+\infty} \theta_0(\lambda) dE_\lambda\right)(\phi) = \phi * \Theta_0 \quad \text{and} \quad \theta_1(\mathcal{J})(\phi) = \left(\int_0^{+\infty} \theta_1(\lambda) dE_\lambda\right)(\phi) = \phi * \Theta_1,$$

where $\phi : \mathbb{G} \longrightarrow \mathbb{R}$ is any function such that these quantities make sense ($\phi \in L^p(\mathbb{G})$ with $1 \leq p < +\infty$ for example). Observe that by construction we have the identity

$$\phi = \phi * \Theta_0 + \phi * \Theta_1, \tag{19}$$

and in particular, applying Proposition 3.1 to the operator's kernel Θ_0 and Θ_1 we can obtain the inequalities

$$\|\Theta_0\|_{L^1} \leq 1 \qquad \text{and} \qquad \|\Theta_1\|_{L^1} \leq 1. \tag{20}$$

Remark 5.1 *We observe that since the function θ_0 is the restriction to $\mathbb{R}^+$ of a smooth function, the kernel Θ_0 associated to the operator $\theta_0(\mathcal{J})$ is also smooth and we have $\|\mathcal{J}^{\frac{s}{2}}\Theta_0\|_{L^1} < +\infty$ for all $0 < s < 1$. However, this is not the case for the kernel Θ_1 which is only integrable : this particular fact will force us to divide our proof in several steps in order to overcome this issue.*

At this stage, we apply the identity (19) to the function $\mathcal{J}^{\frac{s}{2}} f$ in order to obtain

$$\mathcal{J}^{\frac{s}{2}} f = \mathcal{J}^{\frac{s}{2}} f * \Theta_0 + \mathcal{J}^{\frac{s}{2}} f * \Theta_1. \tag{21}$$

Recalling the fact $\mathcal{J}^{\frac{s}{2}} f \in \dot{B}^{-\beta-s,\infty}_{\infty}$, using the thermic definition of Besov spaces and the estimate (17), we obtain the following inequalities, where the integrability properties of the kernels Θ_0 and Θ_1 are used

$$\begin{aligned} \|H_{t_\alpha}\mathcal{J}^{\frac{s}{2}} f * \Theta_0\|_{L^\infty} &\leq \|\Theta_0\|_{L^1}\|H_{t_\alpha}\mathcal{J}^{\frac{s}{2}} f\|_{L^\infty} \leq \|H_{t_\alpha}\mathcal{J}^{\frac{s}{2}} f\|_{L^\infty} \leq \alpha, \\ \text{and} \quad \|H_{t_\alpha}\mathcal{J}^{\frac{s}{2}} f * \Theta_1\|_{L^\infty} &\leq \|\Theta_1\|_{L^1}\|H_{t_\alpha}\mathcal{J}^{\frac{s}{2}} f\|_{L^\infty} \leq \|H_{t_\alpha}\mathcal{J}^{\frac{s}{2}} f\|_{L^\infty} \leq \alpha. \end{aligned} \tag{22}$$

Now, with all these preliminaries, we use the decomposition formula (21) to compute the L^q norm of $\mathcal{J}^{\frac{s}{2}}f$ in the following way

$$\frac{1}{7^q}\|\mathcal{J}^{\frac{s}{2}}f\|_{L^q}^q = q\int_0^{+\infty}\alpha^{q-1}\left|\left\{x\in\mathbb{G}:|\mathcal{J}^{\frac{s}{2}}f(x)|>7\alpha\right\}\right|d\alpha \tag{23}$$
$$= q\int_0^{+\infty}\alpha^{q-1}\left|\left\{x\in\mathbb{G}:|\mathcal{J}^{\frac{s}{2}}f*\Theta_0(x)+\mathcal{J}^{\frac{s}{2}}f*\Theta_1(x)|>7\alpha\right\}\right|d\alpha$$
$$\leq q\int_0^{+\infty}\alpha^{q-1}\left|\left\{x\in\mathbb{G}:|\mathcal{J}^{\frac{s}{2}}f*\Theta_0(x)|>5\alpha\right\}\right|d\alpha + q\int_0^{+\infty}\alpha^{q-1}\left|\left\{x\in\mathbb{G}:|\mathcal{J}^{\frac{s}{2}}f*\Theta_1(x)|>2\alpha\right\}\right|d\alpha$$

We divide our study of these two quantities with the following propositions. The first of these integrals, treated in Proposition 5.1, is easier to study since, as it was pointed out in the Remark 5.1, the kernel Θ_0 is a smooth function and allows us to perform some computations that are not available anymore for the kernel Θ_1. The second integral in the right-hand side of (23) is more technical and it will be treated with Proposition 5.2.

Proposition 5.1 *Under the hypotheses of Theorem 5 and assuming that $\|\mathcal{J}^{\frac{s}{2}}f\|_{L^q}<+\infty$, we have for the first integral of the right-hand side of the expression (23) the inequality*

$$q\int_0^{+\infty}\alpha^{q-1}\left|\left\{x\in\mathbb{G}:|\mathcal{J}^{\frac{s}{2}}f*\Theta_0(x)|>5\alpha\right\}\right|d\alpha\leq Cq\log(M)\|\nabla f\|_{L^1}+\frac{q}{q-1}\frac{1}{M^{q-1}}\|\mathcal{J}^{\frac{s}{2}}f\|_{L^q}^q,$$

where the constant $M\gg 1$ will be fixed later on.

Proof. For the proof of this inequality it is possible to follow with slight modifications the arguments given in [25]. This fact is a consequence of the smoothness properties of the kernel Θ_0. We write then $F=\mathcal{J}^{\frac{s}{2}}f*\Theta_0$ and we introduce the following thresholding function:

$$\Psi_\alpha(t)=\begin{cases}\Psi_\alpha(-t)=-\Psi_\alpha(t),\\ 0 \quad \text{if} \quad 0\leq t\leq\alpha,\\ t-\alpha \quad \text{if} \quad \alpha\leq t\leq M\alpha,\\ (M-1)\alpha \quad \text{if} \quad t>M\alpha,\end{cases}\qquad \text{here, } M \text{ is a parameter which depends on } q.$$

This cut-off function enables us to define a new function $F_\alpha=\Psi_\alpha(F)$ and we write in the next lemma some significant properties of this function F_α:

Lemma 5.1

1) The set defined by $\{x\in\mathbb{G}:|F(x)|>5\alpha\}$ is included in the set $\{x\in\mathbb{G}:|F_\alpha(x)|>4\alpha\}$.

2) On the set $\{x\in\mathbb{G}:|F(x)|\leq M\alpha\}$ one has the estimate $|F-F_\alpha|\leq\alpha$.

3) If $F\in\mathcal{C}^1(\mathbb{G})$, one has the equality $\nabla F_\alpha=(\nabla F)\mathbb{1}_{\{\alpha\leq|F|\leq M\alpha\}}$ almost everywhere.

For a proof see [25]. Thus, by the first point of the lemma above we have

$$q\int_0^{+\infty}\alpha^{q-1}|\{x\in\mathbb{G}:|F(x)|>5\alpha\}|\,d\alpha\leq q\int_0^{+\infty}\alpha^{q-1}|\{x\in\mathbb{G}:|F_\alpha(x)|>4\alpha\}|\,d\alpha. \tag{24}$$

Now, by linearity of the heat semi-group H_{t_α} we can write $F_\alpha=F_\alpha-H_{t_\alpha}(F_\alpha)+H_{t_\alpha}(F_\alpha-F)+H_{t_\alpha}(F)$, and we have

$$\begin{aligned}|\{x\in\mathbb{G}:|F_\alpha(x)|>5\alpha\}| \leq\ & |\{x\in\mathbb{G}:|F_\alpha(x)-H_{t_\alpha}(F_\alpha)(x)|>\alpha\}|+|\{x\in\mathbb{G}:|H_{t_\alpha}(F_\alpha-F)(x)|>2\alpha\}|\\ &+|\{x\in\mathbb{G}:|H_{t_\alpha}(F)(x)|>\alpha\}|,\end{aligned}$$

but $|\{x\in\mathbb{G}:|H_{t_\alpha}(F)(x)|>\alpha\}|=0$ since, by the thermic definition of the Besov space $\dot{B}^{-\beta,\infty}_{\infty}$ we have by (22) the estimate $\|H_{t_\alpha}(F)\|_{L^\infty}\leq\alpha$. Thus returning to (24), we obtain the following inequality

$$\begin{aligned}q\int_0^{+\infty}\alpha^{q-1}|\{x\in\mathbb{G}:|F_\alpha(x)|>4\alpha\}|\,d\alpha \leq\ & q\int_0^{+\infty}\alpha^{q-1}|\{x\in\mathbb{G}:|F_\alpha(x)-H_{t_\alpha}(F_\alpha)(x)|>\alpha\}|\,d\alpha\\ &+q\int_0^{+\infty}\alpha^{q-1}|\{x\in\mathbb{G}:|H_{t_\alpha}(F_\alpha-F)(x)|>2\alpha\}|\,d\alpha.\end{aligned} \tag{25}$$

We will study and estimate these two integrals by the two following lemmas:

Lemma 5.2 *For the first integral of (25) we have the inequality*

$$q\int_0^{+\infty}\alpha^{q-1}\left|\{x\in\mathbb{G}:|F_\alpha(x)-H_{t_\alpha}(F_\alpha)(x)|>\alpha\}\right|d\alpha\leq C\,q\log(M)\|\nabla f\|_{L^1}.$$

Proof. Tchebytchev's inequality implies

$$q\int_0^{+\infty}\alpha^{q-1}\left|\{x\in\mathbb{G}:|F_\alpha(x)-H_{t_\alpha}(F_\alpha)(x)|>\alpha\}\right|d\alpha\leq q\int_0^{+\infty}\alpha^{q-2}\left(\int_{\mathbb{G}}|F_\alpha(x)-H_{t_\alpha}(F_\alpha)(x)|dx\right)d\alpha.$$

We use now the spectral theory to write

$$\begin{aligned}F_\alpha(x)-H_{t_\alpha}(F_\alpha)(x) &= \left(\int_0^{+\infty}(1-e^{-t_\alpha\lambda})dE_\lambda\right)F_\alpha(x)=\left(\int_0^{+\infty}t_\alpha(t_\alpha\lambda)^{-1}(1-e^{-t_\alpha\lambda})\lambda dE_\lambda\right)F_\alpha(x)\\ &= \left(\int_0^{+\infty}t_\alpha m(t_\alpha\lambda)\lambda dE_\lambda\right)F_\alpha(x)=t_\alpha\mathcal{J}\left(F_\alpha*M_{t_\alpha}\right)(x),\end{aligned}$$

where $m(\lambda)=\lambda^{-1}(1-e^{-\lambda})$ defines the operator $m(t_\alpha\mathcal{J})$ which is given by convolution with a kernel M_{t_α} and we have

$$\begin{aligned}q\int_0^{+\infty}\alpha^{q-2}\left(\int_{\mathbb{G}}|F_\alpha(x)-H_{t_\alpha}(F_\alpha)(x)|dx\right)d\alpha &= q\int_0^{+\infty}\alpha^{q-2}\left(\int_{\mathbb{G}}|t_\alpha\mathcal{J}\left(F_\alpha*M_{t_\alpha}\right)(x)|dx\right)d\alpha\\ &= q\int_0^{+\infty}\alpha^{q-2}t_\alpha\left(\int_{\mathbb{G}}|(\nabla F_\alpha*\widetilde{\nabla}M_{t_\alpha})(x)|dx\right)d\alpha\\ &\leq q\int_0^{+\infty}\alpha^{q-2}t_\alpha\left(\int_{\mathbb{G}}|\nabla F_\alpha(x)|dx\right)\|\widetilde{\nabla}M_{t_\alpha}\|_{L^1}d\alpha.\end{aligned}$$

Now, by the properties of the function $m(\lambda)$ and applying Proposition 3.1 we have $\|\widetilde{\nabla}M_{t_\alpha}\|_{L^1}\leq Ct_\alpha^{-1/2}$ and we obtain

$$q\int_0^{+\infty}\alpha^{q-2}\left(\int_{\mathbb{G}}|F_\alpha(x)-H_{t_\alpha}(F_\alpha)(x)|dx\right)d\alpha \leq Cq\int_0^{+\infty}\alpha^{q-2}t_\alpha^{1/2}\left(\int_{\mathbb{G}}|\nabla F_\alpha(x)|dx\right)d\alpha.$$

Remark that the choice of t_α fixed before in (16) gives $t_\alpha^{1/2}=\alpha^{1-q}$, then using Lemma 5.1 we have

$$\begin{aligned}q\int_0^{+\infty}\alpha^{q-2}\left(\int_{\mathbb{G}}|F_\alpha(x)-H_{t_\alpha}(F_\alpha)(x)|dx\right)d\alpha &\leq Cq\int_0^{+\infty}\alpha^{-1}\left(\int_{\{\alpha\leq|F|\leq M\alpha\}}|\nabla F(x)|dx\right)d\alpha\\ &\leq Cq\int_{\mathbb{G}}|\nabla F(x)|\left(\int_{\frac{|F|}{M}}^{|F|}\frac{d\alpha}{\alpha}\right)dx=Cq\log(M)\|\nabla F\|_{L^1}.\end{aligned}$$

It follows then from the definition of F and from the smoothness properties of Θ_0:

$$\begin{aligned}q\int_0^{+\infty}\alpha^{p-2}\left(\int_{\mathbb{G}}|F_\alpha(x)-H_{t_\alpha}(F_\alpha)(x)|dx\right)d\alpha &\leq C\,q\,\log(M)\|\nabla f*\mathcal{J}^{\frac{s}{2}}\Theta_0\|_{L^1}\\ &\leq C\,q\,\log(M)\|\nabla f\|_{L^1}\|\mathcal{J}^{\frac{s}{2}}\Theta_0\|_{L^1}\leq C\,q\,\log(M)\|\nabla f\|_{L^1},\end{aligned}$$

and one obtains the estimation needed for the first integral. ■

Lemma 5.3 *For the second integral of (25) we have the following inequality:*

$$q\int_0^{+\infty}\alpha^{q-1}\left|\{x\in\mathbb{G}:|H_{t_\alpha}(F_\alpha-F)(x)|>2\alpha\}\right|d\alpha\leq\frac{q}{q-1}\,\frac{1}{M^{q-1}}\|\mathcal{J}^{\frac{s}{2}}f\|_{L^q}^q.$$

Proof. For the proof of this lemma, we write

$$|F-F_\alpha|=|F-F_\alpha|\mathbb{1}_{\{|F|\leq M\alpha\}}+|F-F_\alpha|\mathbb{1}_{\{|F|>M\alpha\}},$$

but, by Lemma 5.1, the distance between F and F_α is lower than α on the set $\{x \in \mathbb{G} : |F(x)| \leq M\alpha\}$ and we obtain

$$|F - F_\alpha| \leq \alpha + |F|\mathbb{1}_{\{|F|>M\alpha\}}.$$

Now, applying the heat semi-group to both sides of this inequality we have

$$H_{t_\alpha}(|F - F_\alpha|) \leq \alpha + H_{t_\alpha}(|F|\mathbb{1}_{\{|F|>M\alpha\}}),$$

and then we can write

$$q\int_0^{+\infty} \alpha^{q-1}\left|\{x \in \mathbb{G} : |H_{t_\alpha}(F_\alpha - F)(x)| > 2\alpha\}\right| d\alpha \leq q\int_0^{+\infty} \alpha^{q-1}\left|\{x \in \mathbb{G} : H_{t_\alpha}(|F|\mathbb{1}_{\{|F|>M\alpha\}})(x) > \alpha\}\right| d\alpha.$$

Applying Tchebytchev's inequality, ones has the estimate

$$q\int_0^{+\infty} \alpha^{q-1}\left|\{x \in \mathbb{G} : |H_{t_\alpha}(F_\alpha - F)(x)| > 2\alpha\}\right| d\alpha \leq q\int_0^{+\infty} \alpha^{q-2}\left(\int_{\mathbb{G}} H_{t_\alpha}\left(|F|\mathbb{1}_{\{|F|>M\alpha\}}\right)(x)dx\right)d\alpha,$$

then by Fubini's theorem we have

$$\begin{aligned}
q\int_0^{+\infty} \alpha^{q-1}\left|\{x \in \mathbb{G} : |H_{t_\alpha}(F_\alpha - F)(x)| > 2\alpha\}\right| d\alpha &\leq q\int_{\mathbb{G}} |F(x)|\left(\int_0^{+\infty} \mathbb{1}_{\{|F|>M\alpha\}}\alpha^{q-2}d\alpha\right)dx \\
&\leq \frac{q}{q-1}\int_{\mathbb{G}} |F(x)|\frac{|F(x)|^{q-1}}{M^{q-1}}dx \\
&\leq \frac{q}{q-1}\frac{1}{M^{q-1}}\|F\|^q_{L^q}.
\end{aligned}$$

To finish, we recall that $F = \mathcal{J}^{\frac{s}{2}}f * \Theta_0$ and we have

$$\begin{aligned}
q\int_0^{+\infty} \alpha^{q-1}\left|\{x \in \mathbb{G} : |H_{t_\alpha}(F_\alpha - F)(x)| > 2\alpha\}\right| d\alpha &\leq \frac{q}{q-1}\frac{1}{M^{q-1}}\|\mathcal{J}^{\frac{s}{2}}f * \Theta_0\|^q_{L^q} \\
&\leq \frac{q}{q-1}\frac{1}{M^{q-1}}\|\mathcal{J}^{\frac{s}{2}}f\|^q_{L^q}\|\Theta_0\|^q_{L^1} \\
&\leq \frac{q}{q-1}\frac{1}{M^{q-1}}\|\mathcal{J}^{\frac{s}{2}}f\|^q_{L^q},
\end{aligned}$$

and this concludes the proof of this lemma. ■

We finish the proof of Proposition 5.1 by connecting together these two lemmas *i.e.*:

$$q\int_0^{+\infty} \alpha^{q-1}\left|\left\{x \in \mathbb{G} : |\mathcal{J}^{\frac{s}{2}}f * \Theta_0(x)| > 5\alpha\right\}\right| d\alpha \leq Cq\,\log(M)\|\nabla f\|_{L^1} + \frac{q}{q-1}\frac{1}{M^{q-1}}\|\mathcal{J}^{\frac{s}{2}}f\|^q_{L^q}.$$

■

We continue the proof of Theorem 5 with the following proposition that study the second integral of the right-hand side of (23). This is the most technical part of the proof since we need to deal with the kernel Θ_1, associated to the function θ_1 which does not have the same smoothness properties as the kernel Θ_0.

Proposition 5.2 *Under the hypotheses of Theorem 5, for the second integral of the right-hand side of the expression (23) we have*

$$q\int_0^{+\infty} \alpha^{q-1}\left|\left\{x \in \mathbb{G} : |\mathcal{J}^{\frac{s}{2}}f * \Theta_1(x)| > 2\alpha\right\}\right| d\alpha \leq Cq\|\nabla f\|_{L^1}.$$

Proof. By hypothesis we have $\nabla f \in L^1(\mathbb{G})$ and we will assume that $\|\nabla f\|_{L^1} > 1$. With this extra assumption we can divide our study in three parts. Indeed, for $\sigma > q$ a fixed parameter and denoting by $T = \|\nabla f\|_{L^1}^{\frac{\sigma-1}{\sigma-q}} > 1$, we have

$$\begin{aligned} q\int_0^{+\infty} \alpha^{q-1}\left|\left\{x\in\mathbb{G} : |\mathcal{J}^{\frac{s}{2}}f * \Theta_1(x)| > 2\alpha\right\}\right| d\alpha &= q\int_0^{1} \alpha^{q-1}\left|\left\{x\in\mathbb{G} : |\mathcal{J}^{\frac{s}{2}}f * \Theta_1(x)| > 2\alpha\right\}\right| d\alpha \\ &+q\int_1^{T} \alpha^{q-1}\left|\left\{x\in\mathbb{G} : |\mathcal{J}^{\frac{s}{2}}f * \Theta_1(x)| > 2\alpha\right\}\right| d\alpha \\ &+q\int_T^{+\infty} \alpha^{q-1}\left|\left\{x\in\mathbb{G} : |\mathcal{J}^{\frac{s}{2}}f * \Theta_1(x)| > 2\alpha\right\}\right| d\alpha. \quad (26)\end{aligned}$$

The case when $\|\nabla f\|_{L^1} \leq 1$ is simpler to deal with as we only need to consider the first and the last integral of the right-hand side of the previous expression since, in this particular case, we would have $0 < T < 1$ and then we can write

$$\begin{aligned} q\int_0^{+\infty} \alpha^{q-1}\left|\left\{x\in\mathbb{G} : |\mathcal{J}^{\frac{s}{2}}f * \Theta_1(x)| > 2\alpha\right\}\right| d\alpha &\leq q\int_0^{1} \alpha^{q-1}\left|\left\{x\in\mathbb{G} : |\mathcal{J}^{\frac{s}{2}}f * \Theta_1(x)| > 2\alpha\right\}\right| d\alpha \\ &+q\int_T^{+\infty} \alpha^{q-1}\left|\left\{x\in\mathbb{G} : |\mathcal{J}^{\frac{s}{2}}f * \Theta_1(x)| > 2\alpha\right\}\right| d\alpha,\end{aligned}$$

thus, if we can control the first and the last term of (26) we can also control the previous inequality. As we will see, these two previous integrals will be estimate without the extra hypothesis $\|\nabla f\|_{L^1} > 1$ which is actually only used to study the second term of (26).

Lemma 5.4 *Under the hypotheses of Theorem 5, we have*

$$q\int_0^{1} \alpha^{q-1}\left|\left\{x\in\mathbb{G} : |\mathcal{J}^{\frac{s}{2}}f * \Theta_1(x)| > 2\alpha\right\}\right| d\alpha \leq Cq\|\nabla f\|_{L^1}.$$

Proof. Since, by (22) we have $\|H_{t_\alpha}\mathcal{J}^{\frac{s}{2}}f * \Theta_1\|_{L^\infty} \leq \alpha$ we can write

$$q\int_0^{1} \alpha^{q-1}\left|\left\{x\in\mathbb{G} : |\mathcal{J}^{\frac{s}{2}}f * \Theta_1(x)| > 2\alpha\right\}\right| d\alpha \leq q\int_0^{1} \alpha^{q-1}\left|\left\{x\in\mathbb{G} : |\mathcal{J}^{\frac{s}{2}}f * \Theta_1(x) - H_{t_\alpha}\mathcal{J}^{\frac{s}{2}}f * \Theta_1(x)| > \alpha\right\}\right| d\alpha,$$

and by Tchebychev's inequality we obtain

$$q\int_0^{1} \alpha^{q-1}\left|\left\{x\in\mathbb{G} : |\mathcal{J}^{\frac{s}{2}}f * \Theta_1(x)| > 2\alpha\right\}\right| d\alpha \leq q\int_0^{1} \alpha^{q-2}\left\|\mathcal{J}^{\frac{s}{2}}f * \Theta_1 - H_{t_\alpha}\mathcal{J}^{\frac{s}{2}}f * \Theta_1\right\|_{L^1} d\alpha.$$

We use again the spectral decomposition of the Laplacian to write

$$\mathcal{J}^{\frac{s}{2}}f * \Theta_1 - H_{t_\alpha}\mathcal{J}^{\frac{s}{2}}f * \Theta_1 = \left(\int_0^{+\infty} (1-e^{-t_\alpha\lambda})\lambda^{\frac{s}{2}}\theta_1(\lambda)dE_\lambda\right)(f). \quad (27)$$

The function θ_1 does not vanish at infinity, so in order to perform our computations we need to construct the following decomposition

$$\theta_1(\lambda) = \sum_{j=0}^{+\infty}\psi(2^{-j}\lambda), \quad (28)$$

where ψ is a $\mathcal{C}_0^\infty(\mathbb{R}^+)$ function defined by $\psi(\lambda) = \theta_0(\lambda/2) - \theta_0(\lambda)$. Thus, applying this decomposition to (27) we have

$$\begin{aligned}\mathcal{J}^{\frac{s}{2}}f * \Theta_1 - H_{t_\alpha}\mathcal{J}^{\frac{s}{2}}f * \Theta_1 &= \sum_{j=0}^{+\infty}\left(\int_0^{+\infty}(1-e^{-t_\alpha\lambda})\psi(2^{-j}\lambda)\lambda^{\frac{s}{2}}dE_\lambda\right)(f) \\ &= \sum_{j=0}^{+\infty}\left(\int_0^{+\infty}(2^{-j}\lambda)(1-e^{-t_\alpha\lambda})\psi(2^{-j}\lambda)(2^{-j}\lambda)^{-(1-\frac{s}{2})}2^{j\frac{s}{2}}dE_\lambda\right)(f) \\ &= \sum_{j=0}^{+\infty}\left(\int_0^{+\infty}(2^{-j}\lambda)(1-e^{-t_\alpha\lambda})\widetilde{\psi}(2^{-j}\lambda)2^{j\frac{s}{2}}dE_\lambda\right)(f),\end{aligned}$$

where $\widetilde{\psi}(\lambda) = \frac{\psi(\lambda)}{\lambda^{1-s/2}}$ is a function that belongs to $\mathcal{C}_0^\infty(\mathbb{R}^+)$ and the associated kernel $\widetilde{K}$ belongs to $\mathcal{S}(\mathbb{G})$. We have then that the operator $\widetilde{\psi}(2^{-j}\mathcal{J})$ admits a kernel $\widetilde{K}_j(x) = 2^{j\frac{N}{2}}\widetilde{K}(2^{\frac{j}{2}}x)$ and denoting by M_{t_α} the kernel associated to the operator $m(t_\alpha\mathcal{J})$, where $m(\lambda) = (1 - e^{-\lambda})$, we can write

$$\mathcal{J}^{\frac{s}{2}}f * \Theta_1 - H_{t_\alpha}\mathcal{J}^{\frac{s}{2}}f * \Theta_1 = \sum_{j=0}^{+\infty} 2^{-j(1-s/2)}\mathcal{J}\left(f * \widetilde{K}_j * M_{t_\alpha}\right) = \sum_{j=0}^{+\infty} 2^{-j(1-s/2)}\left(\nabla f * \widetilde{\nabla}\widetilde{K}_j * M_{t_\alpha}\right).$$

With this last identity at hand we can write

$$\begin{aligned} q\int_0^1 \alpha^{q-1}\left|\left\{x \in \mathbb{G} : |\mathcal{J}^{\frac{s}{2}}f * \Theta_1(x)| > 2\alpha\right\}\right| d\alpha &\leq q\int_0^1 \alpha^{q-2}\left\|\sum_{j=0}^{+\infty} 2^{-j(1-s/2)}\left(\nabla f * \widetilde{\nabla}\widetilde{K}_j * M_{t_\alpha}\right)\right\|_{L^1} d\alpha \\ &\leq Cq\int_0^1 \alpha^{q-2}\sum_{j=0}^{+\infty} 2^{-j(1-s/2)}\|\nabla f\|_{L^1}\|\widetilde{\nabla}\widetilde{K}_j\|_{L^1}\|M_{t_\alpha}\|_{L^1} d\alpha. \end{aligned}$$

Here, we apply the Proposition 3.1 to obtain $\|\widetilde{\nabla}\widetilde{K}_j\|_{L^1} \leq C2^{j/2}$ and $\|M_{t_\alpha}\|_{L^1} \leq C$, and thus since $0 < s < 1$ we have

$$\begin{aligned} q\int_0^1 \alpha^{q-1}\left|\left\{x \in \mathbb{G} : |\mathcal{J}^{\frac{s}{2}}f * \Theta_1(x)| > 2\alpha\right\}\right| d\alpha &\leq Cq\|\nabla f\|_{L^1}\int_0^1 \alpha^{q-2}\sum_{j=0}^{+\infty} 2^{-j(1-s/2)}2^{j/2} d\alpha \\ &= Cq\|\nabla f\|_{L^1}\int_0^1 \alpha^{q-2}\sum_{j=0}^{+\infty} 2^{-j(\frac{1-s}{2})} d\alpha \\ &\leq Cq\|\nabla f\|_{L^1}\int_0^1 \alpha^{q-2} d\alpha \leq Cq\|\nabla f\|_{L^1}, \end{aligned}$$

and the Lemma 5.4 is proven. ■

It is worth noting that in the proof of this lemma, we do not need the extra assumption $\|\nabla f\|_{L^1} > 1$, this hypothesis will be useful in the following lemma which is the most technical part of the proof of Theorem 5.

Lemma 5.5 *Under the hypotheses of Theorem 5 and assuming that $\|\nabla f\|_{L^1} > 1$, we have for the second integral of the right-hand side of (26) the following inequality*

$$q\int_1^T \alpha^{q-1}\left|\left\{x \in \mathbb{G} : |\mathcal{J}^{\frac{s}{2}}f * \Theta_1(x)| > 2\alpha\right\}\right| d\alpha \leq Cq\|\nabla f\|_{L^1}.$$

Proof. Since, by (22) we have $\|H_{t_\alpha}\mathcal{J}^{\frac{s}{2}}f * \Theta_1\|_{L^\infty} \leq \alpha$ we can write

$$q\int_1^T \alpha^{q-1}\left|\left\{x \in \mathbb{G} : |\mathcal{J}^{\frac{s}{2}}f * \Theta_1(x)| > 2\alpha\right\}\right| d\alpha \leq q\int_1^T \alpha^{q-1}\left|\left\{x \in \mathbb{G} : |\mathcal{J}^{\frac{s}{2}}f * \Theta_1(x) - H_{t_\alpha}\mathcal{J}^{\frac{s}{2}}f * \Theta_1(x)| > \alpha\right\}\right| d\alpha,$$

and by Tchebychev's inequality we obtain

$$q\int_1^T \alpha^{q-1}\left|\left\{x \in \mathbb{G} : |\mathcal{J}^{\frac{s}{2}}f * \Theta_1(x)| > 2\alpha\right\}\right| d\alpha \leq q\int_1^T \alpha^{q-2}\left\|\mathcal{J}^{\frac{s}{2}}f * \Theta_1 - H_{t_\alpha}\mathcal{J}^{\frac{s}{2}}f * \Theta_1\right\|_{L^1} d\alpha. \tag{29}$$

We use again the spectral decomposition of the Laplacian to write

$$\mathcal{J}^{\frac{s}{2}}f * \Theta_1 - H_{t_\alpha}\mathcal{J}^{\frac{s}{2}}f * \Theta_1 = \left(\int_0^{+\infty}(1 - e^{-t_\alpha\lambda})\lambda^{\frac{s}{2}}\theta_1(\lambda)dE_\lambda\right)(f). \tag{30}$$

Here we will need another decomposition and we can not simply repeat the decomposition used previously in (28). We start by writing $\varepsilon = \varepsilon(\alpha) = \alpha^{-2\delta}$ with $\delta > \frac{p-1}{1-s}$. Remark that the integration over α in (29) runs between 1 and T so we have that $\varepsilon \in]T^{-2\delta}, 1]$. Then we define a new auxiliary function ϕ_ε by the expression

$$\phi_\varepsilon(\lambda) = \theta_0(2^{-2}\varepsilon\lambda) - \theta_0(\lambda), \tag{31}$$

and this is a $\mathcal{C}_0^\infty(\mathbb{R}^+)$ function. Now, for $j \geq 0$ we define

$$\eta(2^{-j}\varepsilon\lambda) = \theta_0(2^{-(j+1)}2^{-2}\varepsilon\lambda) - \theta_0(2^{-j}2^{-2}\varepsilon\lambda),$$

and we obtain the following decomposition

$$\theta_1(\lambda) = \phi_\varepsilon(\lambda) + \sum_{j=0}^{+\infty}\eta(2^{-j}\varepsilon\lambda).$$

It is important to remark that this decomposition of the function θ_1 contains two terms of different nature. Indeed, the function η is in some sense homogeneous since by construction it has the same decay for lower or higher values of λ and the corresponding kernel will satisfy homogeneous properties as in Proposition 3.1, but this is not the case for the term ϕ_ε as for lowers values of λ it has a different decay than for higher values of λ (this is clear from the formula (31)) and we can not apply directly the ideas behind the Proposition 3.1 to the kernel associated to the function ϕ_ε. This particular feature will force us to treat in a separate way each of these terms.

Having in mind this remark, we apply this decomposition to (30) and we have

$$\mathcal{J}^{\frac{s}{2}}f * \Theta_1 - H_{t_\alpha}\mathcal{J}^{\frac{s}{2}}f * \Theta_1 = \left(\int_0^{+\infty}(1-e^{-t_\alpha\lambda})\phi_\varepsilon(\lambda)\lambda^{\frac{s}{2}}dE_\lambda\right)(f) + \sum_{j=0}^{+\infty}\left(\int_0^{+\infty}(1-e^{-t_\alpha\lambda})\eta(2^{-j}\varepsilon\lambda)\lambda^{\frac{s}{2}}dE_\lambda\right)(f),$$

then we obtain the following estimate for the right-hand side of (29):

$$\begin{aligned} q\int_1^T \alpha^{q-2}\left\|\mathcal{J}^{\frac{s}{2}}f * \Theta_1 - H_{t_\alpha}\mathcal{J}^{\frac{s}{2}}f * \Theta_1\right\|_{L^1}d\alpha &\leq q\int_1^T \alpha^{q-2}\left\|\left(\int_0^{+\infty}(1-e^{-t_\alpha\lambda})\phi_\varepsilon(\lambda)\lambda^{\frac{s}{2}}dE_\lambda\right)(f)\right\|_{L^1}d\alpha \\ &+ q\int_1^T \alpha^{q-2}\left\|\sum_{j=0}^{+\infty}\left(\int_0^{+\infty}(1-e^{-t_\alpha\lambda})\eta(2^{-j}\varepsilon\lambda)\lambda^{\frac{s}{2}}dE_\lambda\right)(f)\right\|_{L^1}d\alpha \end{aligned} \tag{32}$$

As said in the lines above, we need to study separately each one of the terms in the right-hand side of the previous inequality (32). For the first part of this formula we will prove the following inequality:

$$q\int_1^T \alpha^{q-2}\left\|\left(\int_0^{+\infty}(1-e^{-t_\alpha\lambda})\phi_\varepsilon(\lambda)\lambda^{\frac{s}{2}}dE_\lambda\right)(f)\right\|_{L^1}d\alpha \leq Cq\|\nabla f\|_{L^1}. \tag{33}$$

Indeed, since we have $1 < T = \|\nabla f\|_{L^1}^{\frac{\sigma-1}{\sigma-q}} < +\infty$, there exists an integer N_T such that $T^{2\delta} < 2^{N_T}$. We define thus a new auxiliary function given by the expression

$$\Phi(\lambda) = \theta_0(2^{-N_T-2}\lambda) - \theta_0(2\lambda),$$

since by construction $\Phi = 1$ in the support of ϕ_ε for all $\varepsilon \in]T^{-2\delta}, 1]$, we obtain the identity

$$\phi_\varepsilon(\lambda) = \Phi(\lambda)\phi_\varepsilon(\lambda),$$

and then, we have

$$\left(\int_0^{+\infty}(1-e^{-t_\alpha\lambda})\phi_\varepsilon(\lambda)\lambda^{\frac{s}{2}}dE_\lambda\right)(f) = \left(\int_0^{+\infty}(1-e^{-t_\alpha\lambda})\Phi(\lambda)\phi_\varepsilon(\lambda)\lambda^{\frac{s}{2}}dE_\lambda\right)(f).$$

Here we will perform some computations at the spectral level

$$\begin{aligned} \left(\int_0^{+\infty}(1-e^{-t_\alpha\lambda})\phi_\varepsilon(\lambda)\lambda^{\frac{s}{2}}dE_\lambda\right)(f) &= \left(\int_0^{+\infty}t_\alpha(t_\alpha\lambda)^{-1}(1-e^{-t_\alpha\lambda})\lambda^{\frac{s}{2}}\Phi(\lambda)\phi_\varepsilon(\lambda)\,\lambda dE_\lambda\right)(f) \\ &= \left(\int_0^{+\infty}t_\alpha\Gamma(t_\alpha\lambda)\widetilde{\Phi}(\lambda)\phi_\varepsilon(\lambda)\lambda dE_\lambda\right)(f), \end{aligned}$$

where we noted $\Gamma(t_\alpha\lambda) = (t_\alpha\lambda)^{-1}(1 - e^{-t_\alpha\lambda})$ and $\widetilde{\Phi}(\lambda) = \lambda^{\frac{s}{2}}\Phi(\lambda)$. If we consider now the associated operators to these functions we have that $\Gamma(t_\alpha\mathcal{J})$, $\widetilde{\Phi}(\mathcal{J})$ and $\phi_\varepsilon(\mathcal{J})$ admit a convolution kernel denoted respectively by Π_{t_α}, K and Σ_ε. Thus, we obtain the following expression

$$\left(\int_0^{+\infty}(1 - e^{-t_\alpha\lambda})\phi_\varepsilon(\lambda)\lambda^{\frac{s}{2}}dE_\lambda\right)(f) \quad = \quad t_\alpha\,\mathcal{J}\left(f * \Pi_{t_\alpha} * K * \Sigma_\varepsilon\right) = t_\alpha\,\left(\nabla f * \widetilde{\nabla}\Pi_{t_\alpha} * K * \Sigma_\varepsilon\right).$$

At this point, in order to obtain formula (33) we first take the L^1 norm of the previous expression to obtain

$$\left\|\left(\int_0^{+\infty}(1 - e^{-t_\alpha\lambda})\phi_\varepsilon(\lambda)\lambda^{\frac{s}{2}}dE_\lambda\right)(f)\right\|_{L^1} \quad \leq \quad t_\alpha\|\nabla f\|_{L^1}\|\widetilde{\nabla}\Pi_{t_\alpha}\|_{L^1}\|K\|_{L^1}\|\Sigma_\varepsilon\|_{L^1}.$$

Now, by the properties of the auxiliary functions and applying Proposition 3.1 to these kernels we obtain the following inequalities $\|\widetilde{\nabla}\Pi_{t_\alpha}\|_{L^1} \leq Ct_\alpha^{-1/2}$, $\|K\|_{L^1} \leq C$ and $\|\Sigma_\varepsilon\|_{L^1} \leq C$. Thus we can write

$$\left\|\left(\int_0^{+\infty}(1 - e^{-t_\alpha\lambda})\phi_\varepsilon(\lambda)\lambda^{\frac{s}{2}}dE_\lambda\right)(f)\right\|_{L^1} \quad \leq \quad Ct_\alpha^{1/2}\|\nabla f\|_{L^1}.$$

It is important to remark here that due to the Proposition 3.1, since the functions $\Gamma, \widetilde{\Phi}$ and ϕ_ε are bounded, the constant C depends only on the dimension and eventually on other parameters associated to the underlying group structure but it is independent from the parameters ε and T.

Once we have this inequality, we integrate it with respect to α in order to obtain

$$q\int_1^T \alpha^{q-2}\left\|\left(\int_0^{+\infty}(1 - e^{-t_\alpha\lambda})\phi_\varepsilon(\lambda)\lambda^{\frac{s}{2}}dE_\lambda\right)(f)\right\|_{L^1} d\alpha \quad \leq \quad Cq\|\nabla f\|_{L^1}\int_1^T \alpha^{q-2}t_\alpha^{1/2}d\alpha.$$

Since $t_\alpha = \alpha^{-\frac{2}{\beta+s}} = \alpha^{-\frac{2(q-1)}{1-s}}$, we have

$$\begin{aligned}
q\int_1^T \alpha^{q-2}\left\|\left(\int_0^{+\infty}(1 - e^{-t_\alpha\lambda})\phi_\varepsilon(\lambda)\lambda^{\frac{s}{2}}dE_\lambda\right)(f)\right\|_{L^1} d\alpha \quad &\leq \quad Cq\|\nabla f\|_{L^1}\int_1^T \alpha^{q-2}\alpha^{-\frac{q-1}{1-s}}d\alpha \\
&\leq \quad Cq\|\nabla f\|_{L^1}\int_1^T \alpha^{-(q-1)(\frac{s}{1-s})-1}d\alpha \\
&\leq \quad Cq\|\nabla f\|_{L^1}\left(1 - T^{-(q-1)(\frac{s}{1-s})}\right),
\end{aligned}$$

and since $T > 1$, we finally have

$$q\int_1^T \alpha^{q-2}\left\|\left(\int_0^{+\infty}(1 - e^{-t_\alpha\lambda})\phi_\varepsilon(\lambda)\lambda^{\frac{s}{2}}dE_\lambda\right)(f)\right\|_{L^1} d\alpha \leq Cq\|\nabla f\|_{L^1}. \tag{34}$$

Now, we will study the second part of the right-hand side of the expression (32) and we will prove the following inequality

$$q\int_1^T \alpha^{q-2}\left\|\sum_{j=0}^{+\infty}\left(\int_0^{+\infty}(1 - e^{-t_\alpha\lambda})\eta(2^{-j}\varepsilon\lambda)\lambda^{\frac{s}{2}}dE_\lambda\right)(f)\right\|_{L^1} d\alpha \leq Cq\|\nabla f\|_{L^1}.$$

This inequality will be easier to obtain than the previous one given by (34) since the function η satisfies suitable homogeneous properties which was not the case for the function ϕ_ε. Indeed, we observe that

$$\sum_{j=0}^{+\infty}\left(\int_0^{+\infty}(2^{-j}\varepsilon\lambda)(1 - e^{-t_\alpha\lambda})\eta(2^{-j}\varepsilon\lambda)(2^{-j}\varepsilon\lambda)^{-1}\lambda^{\frac{s}{2}}dE_\lambda\right)(f) = \sum_{j=0}^{+\infty}\left(\int_0^{+\infty}(2^{-j}\varepsilon)^{1-s/2}(1 - e^{-t_\alpha\lambda})\widetilde{\eta}(2^{-j}\varepsilon\lambda)\,\lambda dE_\lambda\right)(f),$$

where $\widetilde{\eta}(\lambda) = \frac{\eta(\lambda)}{\lambda^{1-s/2}}$ is a function that belongs to $\mathcal{C}_0^\infty(\mathbb{R}^+)$ and the kernel $\widetilde{E}$ associated to the operator $\widetilde{\eta}(\mathcal{J})$ belongs to $\mathcal{S}(\mathbb{G})$ and we have then that the operator $\widetilde{\eta}(2^{-j}\varepsilon\mathcal{J})$ admits a kernel $\widetilde{E}_{j,\varepsilon}(x) = (2^{-j}\varepsilon)^{-\frac{N}{2}}\widetilde{E}((2^{-j}\varepsilon)^{-\frac{1}{2}}x)$. Furthermore, if we denote by M_{t_α} the kernel associated to the operator $m(t_\alpha\mathcal{J})$ where $m(\lambda)=(1-e^{-\lambda})$, we can write

$$\begin{aligned}\sum_{j=0}^{+\infty}\left(\int_0^{+\infty}(1-e^{-t_\alpha\lambda})\eta(2^{-j}\varepsilon\lambda)\lambda^{\frac{s}{2}}dE_\lambda\right)(f) &= \sum_{j=0}^{+\infty}(2^{-j}\varepsilon)^{1-s/2}\mathcal{J}\left(f*\widetilde{E}_{j,\varepsilon}*M_{t_\alpha}\right)\\ &= \sum_{j=0}^{+\infty}(2^{-j}\varepsilon)^{1-s/2}\left(\nabla f*\widetilde{\nabla}\widetilde{E}_{j,\varepsilon}*M_{t_\alpha}\right).\end{aligned}$$

With this last identity at hand we obtain

$$q\int_1^T\alpha^{q-2}\left\|\sum_{j=0}^{+\infty}\left(\int_0^{+\infty}(1-e^{-t_\alpha\lambda})\eta(2^{-j}\varepsilon\lambda)\lambda^{\frac{s}{2}}dE_\lambda\right)(f)\right\|_{L^1}d\alpha\leq q\int_1^T\alpha^{q-2}\left\|\sum_{j=0}^{+\infty}(2^{-j}\varepsilon)^{1-s/2}\left(\nabla f*\widetilde{\nabla}\widetilde{E}_{j,\varepsilon}*M_{t_\alpha}\right)\right\|_{L^1}d\alpha$$

$$\leq Cq\int_1^T\alpha^{q-2}\sum_{j=0}^{+\infty}(2^{-j}\varepsilon)^{1-s/2}\|\nabla f\|_{L^1}\|\widetilde{\nabla}\widetilde{E}_{j,\varepsilon}\|_{L^1}\|M_{t_\alpha}\|_{L^1}d\alpha.$$

Now, we apply Proposition 3.1 to obtain that $\|\widetilde{\nabla}\widetilde{E}_{j,\varepsilon}\|_{L^1}\leq C(2^{-j}\varepsilon)^{-1/2}$ and $\|M_{t_\alpha}\|_{L^1}\leq C$. Thus, as $0<s<1$, we have the following inequality

$$q\int_1^T\alpha^{q-2}\left\|\sum_{j=0}^{+\infty}\left(\int_0^{+\infty}(1-e^{-t_\alpha\lambda})\eta(2^{-j}\varepsilon\lambda)\lambda^{\frac{s}{2}}dE_\lambda\right)(f)\right\|_{L^1}d\alpha\leq Cq\|\nabla f\|_{L^1}\int_1^T\alpha^{q-2}\sum_{j=0}^{+\infty}(2^{-j}\varepsilon)^{1-s/2}(2^{-j}\varepsilon)^{-1/2}d\alpha$$

$$\leq Cq\|\nabla f\|_{L^1}\int_1^T\alpha^{q-2}\sum_{j=0}^{+\infty}2^{-j(\frac{1-s}{2})}\varepsilon^{\frac{1-s}{2}}d\alpha\leq Cq\|\nabla f\|_{L^1}\int_1^T\alpha^{p-2}\varepsilon^{\frac{1-s}{2}}d\alpha,$$

but, since $\varepsilon=\varepsilon(\alpha)=\alpha^{-2\delta}$ with $\delta>\frac{q-1}{1-s}$ and since we assumed that $T>1$, we have

$$q\int_1^T\alpha^{q-2}\left\|\sum_{j=0}^{+\infty}\left(\int_0^{+\infty}(1-e^{-t_\alpha\lambda})\eta(2^{-j}\varepsilon\lambda)\lambda^{\frac{s}{2}}dE_\lambda\right)(f)\right\|_{L^1}d\alpha\leq Cq\|\nabla f\|_{L^1}\int_1^T\alpha^{q-2}\alpha^{-\delta(1-s)}d\alpha$$

$$\leq Cq\|\nabla f\|_{L^1}\left(1-T^{-(\delta(1-s)-(q-1))}\right)\leq Cq\|\nabla f\|_{L^1}.$$

Remark that the quantity in brackets above is always bounded by 1, thus the constant C is an universal constant which is independent from the parameter T.

Finally, with this inequality for the second part of the right-hand side of (32) and with the previous inequality (34) for the first part, we have that

$$q\int_1^T\alpha^{q-2}\left\|\mathcal{J}^{\frac{s}{2}}f*\Theta_1-H_{t_\alpha}\mathcal{J}^{\frac{s}{2}}f*\Theta_1\right\|_{L^1}d\alpha\quad\leq\quad Cq\|\nabla f\|_{L^1},$$

and this concludes the proof of the Lemma 5.5. ■

Lemma 5.6 *Under the hypotheses of Theorem 5, we have for the last integral of the right-hand side of (26) the following inequality*

$$q\int_T^{+\infty}\alpha^{q-1}\left|\left\{x\in\mathbb{G}:|\mathcal{J}^{\frac{s}{2}}f*\Theta_1(x)|>2\alpha\right\}\right|d\alpha\leq C\|\nabla f\|_{L^1}.$$

Proof. If $\sigma>q$, applying Tchebychev's inequality we have

$$q\int_T^{+\infty}\alpha^{q-1}\left|\left\{x\in\mathbb{G}:|\mathcal{J}^{\frac{s}{2}}f*\Theta_1(x)|>2\alpha\right\}\right|d\alpha\leq q\int_T^{+\infty}\alpha^{q-1-\sigma}\|\mathcal{J}^{\frac{s}{2}}f*\Theta_1\|_{L^\sigma}^\sigma d\alpha.$$

Now, by Young's inequality in weak L^σ spaces given in Lemma 4.1 we have

$$q\int_T^{+\infty}\alpha^{q-1}\left|\left\{x\in\mathbb{G}:|\mathcal{J}^{\frac{s}{2}}f*\Theta_1(x)|>2\alpha\right\}\right|d\alpha\leq q\int_T^{+\infty}\alpha^{q-1-\sigma}\|\mathcal{J}^{\frac{s}{2}}f\|_{L^{q,\infty}}^\sigma\|\Theta_1\|_{L^r}^\sigma d\alpha,$$

where $1+\frac{1}{\sigma}=\frac{1}{q}+\frac{1}{r}$. Since by Proposition 3.1 we have $\|\Theta_1\|_{L^r}\leq C$, using the weak Sobolev inequalities (4) we obtain the following inequality

$$q\int_T^{+\infty}\alpha^{q-1}\left|\left\{x\in\mathbb{G}:|\mathcal{J}^{\frac{s}{2}}f*\Theta_1(x)|>2\alpha\right\}\right|d\alpha \quad\leq\quad Cq\|\mathcal{J}^{\frac{s}{2}}f\|_{L^{p,\infty}}^\sigma\int_T^{+\infty}\alpha^{q-1-\sigma}d\alpha\leq Cq\|\nabla f\|_{L^1}^{\sigma/q}\|f\|_{\dot{B}^{-\beta,\infty}_\infty}^{\sigma(1-1/q)}T^{-(\sigma-q)}.$$

But since we assumed that $\|f\|_{\dot{B}^{-\beta,\infty}_\infty}\leq 1$ (up to a normalization constant since $\|f\|_{\dot{B}^{-\beta,\infty}_\infty}\simeq\|\mathcal{J}^{\frac{s}{2}}f\|_{\dot{B}^{-\beta-s,\infty}_\infty}\leq 1$), by the definition of $T=\|\nabla f\|_{L^1}^{\frac{\sigma-1}{\sigma-q}}$ we finally obtain

$$q\int_T^{+\infty}\alpha^{q-1}\left|\left\{x\in\mathbb{G}:|\mathcal{J}^{\frac{s}{2}}f*\Theta_1(x)|>2\alpha\right\}\right|d\alpha\leq Cq\|\nabla f\|_{L^1},$$

and the proof of the Lemma 5.6 is finished. ■

Now, with Lemmas 5.4, 5.5 and 5.6, we can come back to (26) and we obtain the following inequality:

$$\begin{aligned}q\int_0^{+\infty}\alpha^{q-1}\left|\left\{x\in\mathbb{G}:|\mathcal{J}^{\frac{s}{2}}f*\Theta_1(x)|>2\alpha\right\}\right|d\alpha \quad&=\quad q\int_0^{1}\alpha^{q-1}\left|\left\{x\in\mathbb{G}:|\mathcal{J}^{\frac{s}{2}}f*\Theta_1(x)|>2\alpha\right\}\right|d\alpha\\&\quad\;+q\int_1^{T}\alpha^{q-1}\left|\left\{x\in\mathbb{G}:|\mathcal{J}^{\frac{s}{2}}f*\Theta_1(x)|>2\alpha\right\}\right|d\alpha\\&\quad\;+q\int_T^{+\infty}\alpha^{q-1}\left|\left\{x\in\mathbb{G}:|\mathcal{J}^{\frac{s}{2}}f*\Theta_1(x)|>2\alpha\right\}\right|d\alpha\\&\leq\quad Cq\|\nabla f\|_{L^1},\end{aligned}$$

which is the conclusion of Proposition 5.2. ■

We have proven Proposition 5.1 and Proposition 5.2 and we continue now the proof of Theorem 5. With these inequalities at hand, we can return to the inequality (23) and we obtain

$$\frac{1}{7^q}\|\mathcal{J}^{\frac{s}{2}}f\|_{L^q}^q\leq q\int_0^{+\infty}\alpha^{q-1}\left|\left\{x\in\mathbb{G}:|\mathcal{J}^{\frac{s}{2}}f*\Theta_0(x)|>5\alpha\right\}\right|d\alpha+q\int_0^{+\infty}\alpha^{q-1}\left|\left\{x\in\mathbb{G}:|\mathcal{J}^{\frac{s}{2}}f*\Theta_1(x)|>2\alpha\right\}\right|d\alpha$$

$$\leq Cq\log(M)\|\nabla f\|_{L^1}+\frac{q}{q-1}\frac{1}{M^{q-1}}\|\mathcal{J}^{\frac{s}{2}}f\|_{L^q}^q+Cq\|\nabla f\|_{L^1}.$$

Now, if M is constant big enough, as we assumed $\|\mathcal{J}^{\frac{s}{2}}f\|_{L^q}<+\infty$ we can write

$$\left(\frac{1}{7^q}-\frac{q}{q-1}\frac{1}{M^{q-1}}\right)\|\mathcal{J}^{\frac{s}{2}}f\|_{L^q}^q\leq Cq\log(M)\|\nabla f\|_{L^1}.$$

However, the proof of the Theorem 5 is not complete since we worked with the extra condition that $\|\mathcal{J}^{\frac{s}{2}}f\|_{L^q}<+\infty$. To overcome this issue we proceed as follows.

Proposition 5.3 *It is possible to consider only the two assumptions $\nabla f\in L^1(\mathbb{G})$ and $f\in\dot{B}^{-\beta,\infty}_\infty(\mathbb{G})$ in order to obtain the inequality*

$$\|f\|_{\dot{W}^{s,q}}\leq C\|\nabla f\|_{L^1}^\theta\|f\|_{\dot{B}^{-\beta,\infty}_\infty}^{1-\theta},$$

where $1<q<+\infty$, $0\leq s<1/q$, $\beta=\frac{1-sq}{q-1}$ and $\theta=\frac{1}{q}$.

Proof. For the proof of this proposition we use again the spectral theory to build an approximation of f in the following way

$$f_j = \left(\int_0^{+\infty} \big(\varphi(2^{-2j}\lambda) - \varphi(2^{2j}\lambda)\big)\, dE_\lambda\right)(f) \qquad (j \in \mathbb{N}),$$

where φ is a $\mathcal{C}^\infty(\mathbb{R}^+)$ function such that $\varphi = 1$ on $]0, 1/4[$ and $\varphi = 0$ on $[1, +\infty[$.

Lemma 5.7 *If $q > 1$, if $\nabla f \in L^1(\mathbb{G})$ and if $f \in \dot{B}^{-\beta,\infty}_{\infty}(\mathbb{G})$ then $\nabla f_j \in L^1(\mathbb{G})$, $f_j \in \dot{B}^{-\beta,\infty}_{\infty}(\mathbb{G})$ and $f_j \in L^q(\mathbb{G})$.*

Proof. The fact that $\nabla f_j \in L^1(\mathbb{G})$ and $f_j \in \dot{B}^{-\beta,\infty}_{\infty}(\mathbb{G})$ is an easy consequence of the definition of f_j. To prove that $f_j \in L^q(\mathbb{G})$ we will use the identity

$$f_j = \left(\int_0^{+\infty} m(2^{-2j}\lambda)\, dE_\lambda\right) 2^{-2j}\mathcal{J}(f),$$

where we noted

$$m(2^{-2j}\lambda) = \frac{\varphi(2^{-2j}\lambda) - \varphi(2^{2j}\lambda)}{2^{-2j}\lambda}.$$

Observe that the function m is in $\mathcal{C}^\infty(\mathbb{R}^+)$, we obtain then the following identity where $M_j \in \mathcal{S}(\mathbb{G})$ is the kernel of the operator $m(2^{-2j}\mathcal{J})$

$$f_j = 2^{-2j}\mathcal{J} f * M_j = 2^{-2j}\nabla f * \widetilde{\nabla} M_j.$$

Taking the L^q norm in the preceding expression we have

$$\|f_j\|_{L^q} = \|2^{-2j}\nabla f * \widetilde{\nabla} M_j\|_{L^q} \leq 2^{-2j}\|\nabla f\|_{L^1}\|\widetilde{\nabla} M_j\|_{L^q}.$$

Finally, applying Proposition 3.1 to the norm $\|\widetilde{\nabla} M_j\|_{L^q}$ we obtain:

$$\|f_j\|_{L^q} \leq C\, 2^{j(N(1-\frac{1}{q})-1)}\|\nabla f\|_{L^1} < +\infty.$$

■

Thanks to this estimate, we can apply the previous computations made with the Propositions 5.1 and 5.2 to the functions f_j whose L^q norm is bounded, and we obtain the inequality

$$\|f_j\|_{L^q} \leq C\|\nabla f_j\|^{\theta}_{L^1}\|f_j\|^{1-\theta}_{\dot{B}^{-\beta,\infty}_{\infty}}.$$

Now, since $f \in \dot{B}^{-\beta,\infty}_{\infty}(\mathbb{G})$, we have $f_j \rightharpoonup f$ in the sense of distributions. It follows

$$\|f\|_{L^q} \leq \liminf_{j\to+\infty}\|f_j\|_{L^q} \leq C\|\nabla f\|^{\theta}_{L^1}\|f\|^{1-\theta}_{\dot{B}^{-\beta,\infty}_{\infty}}.$$

We have restricted ourselves to the two initial assumptions, namely $\nabla f \in L^1(\mathbb{G})$ and $f \in \dot{B}^{-\beta,\infty}_{\infty}(\mathbb{G})$. Theorem 5 is now completely proven.

■

6 Proof of Theorem 2

We will prove here, in the framework of stratified Lie groups, the inequality

$$\|f\|_{\dot{\Lambda}^{s_1,q}(w)} \leq C\|f\|^{\theta}_{\dot{\Lambda}^{s,p}(w)}\|f\|^{1-\theta}_{\dot{B}^{-\beta,\infty}_{\infty}},$$

where $f : \mathbb{G} \longrightarrow \mathbb{R}$ is a function such that $f \in \dot{\Lambda}^{s,p}(w)(\mathbb{G}) \cap \dot{B}^{-\beta,\infty}_{\infty}(\mathbb{G})$ with $1 < p < q < +\infty$, $\theta = p/q$, $s_1 = \theta s - (1-\theta)\beta$ and $-\beta < s_1 < s$. We will always assume here that w is a weight in the Ariño-Muckenhoupt class B_p. The reason for this particular choice of weights relies on the fact that we will need the boundedness of the Hardy-Littlewood maximal operator on Lorentz $\Lambda^p(w)$ spaces and this is ensured by the condition $w \in B_p$. See [3] and [10] for details.

By the definition of Lorentz-Sobolev spaces given in Section 4, this inequality can be rewritten in the following way

$$\|\mathcal{J}^{\frac{s_1}{2}} f\|_{\Lambda^q(w)} \leq C\|\mathcal{J}^{\frac{s}{2}} f\|^{\theta}_{\Lambda^p(w)}\|f\|^{1-\theta}_{\dot{B}^{-\beta,\infty}_{\infty}}.$$

For the proof of this inequality, we will use a variant of Hedberg's inequality. Indeed, since $0 < s_1 < s$, we use the characterization of the positive powers of the Laplacian given in (12) and we have for $k > s/2 > s_1/2$

$$\begin{aligned}\mathcal{J}^{\frac{s_1}{2}} f(x) &= \frac{1}{\Gamma(k-s_1/2)}\int_0^{+\infty} t^{k-\frac{s_1}{2}-1}\mathcal{J}^k H_t f(x)dt\\ &= \frac{1}{\Gamma(k-s_1/2)}\left(\int_0^{T} t^{k-\frac{s_1}{2}-1}\mathcal{J}^k H_t f(x)dt + \int_T^{+\infty} t^{k-\frac{s_1}{2}-1}\mathcal{J}^k H_t f(x)dt\right),\end{aligned}$$

where T will be defined below. In particular we have

$$|\mathcal{J}^{\frac{s_1}{2}} f(x)| \leq \frac{1}{\Gamma(k-s_1/2)}\left(\int_0^{T} t^{k-\frac{s_1}{2}-1}|\mathcal{J}^k H_t f(x)|dt + \int_T^{+\infty} t^{k-\frac{s_1}{2}-1}|\mathcal{J}^k H_t f(x)|dt\right). \tag{35}$$

For the first integral of the right-hand side of the previous formula we will use the following fact.

Lemma 6.1 *Let $f \in \mathcal{S}'(\mathbb{G})$ and $\varphi \in \mathcal{S}(\mathbb{G})$. We denote by $\mathcal{M}_\varphi(f)$ the maximal function of f (with respect to φ) which is given by the expression*

$$\mathcal{M}_\varphi f(x) = \sup_{0<t<+\infty}\{|f * \varphi_t(x)|\}, \quad \text{with } \varphi_t(x) = t^{-N/2}\varphi(t^{-1/2}x).$$

If the function φ is such that $|\varphi(x)| \leq C(1+|x|)^{-N-\varepsilon}$ for some $\varepsilon > 0$, then we have the following pointwise inequality

$$\mathcal{M}_\varphi f(x) \leq C\mathcal{M}_B f(x),$$

where $\mathcal{M}_B f(x)$ is the Hardy-Littlewood maximal function defined by (13).

For a proof of this lemma see [22] or [18]. With this lemma in mind, and since $k > s/2$, we remark that we have the identity

$$\mathcal{J}^k H_t f(x) = \mathcal{J}^{k-\frac{s}{2}} h_t * \mathcal{J}^{\frac{s}{2}} f(x).$$

Now, by homogeneity we obtain $\mathcal{J}^{k-\frac{s}{2}}(h_t)(x) = t^{-k+\frac{s}{2}}\big(\mathcal{J}^{k-\frac{s}{2}}h_t\big)(x)$ and if we denote φ_t by $\varphi_t(x) = \big(\mathcal{J}^{k-\frac{s}{2}}h_t\big)(x)$ we have that $\varphi_t(x) = t^{-N/2}\varphi(t^{-1/2}x)$, moreover, since the heat kernel h_t is a smooth function, with the previous notation we obtain $|\varphi(x)| \leq C(1+|x|)^{-N-\varepsilon}$. Then we can write

$$\mathcal{J}^k H_t f(x) = t^{-k+\frac{s}{2}}\varphi_t * \mathcal{J}^{\frac{s}{2}} f(x),$$

and applying the Lemma 6.1 we have the following pointwise inequality for the first term of (35):

$$|\mathcal{J}^k H_t f(x)| = t^{-k+\frac{s}{2}}\mathcal{M}_B\left(\mathcal{J}^{\frac{s}{2}} f\right)(x).$$

Now, for the second integral of the right-hand side of (35) we simply use the fact that $\|\mathcal{J}^k f\|_{\dot{B}^{-\beta-2k,\infty}_{\infty}} \simeq \|f\|_{\dot{B}^{-\beta,\infty}_{\infty}}$ and the thermic definition of Besov spaces to obtain

$$|\mathcal{J}^k H_t f(x)| = |H_t\mathcal{J}^k f(x)| \leq Ct^{\frac{-\beta-2k}{2}}\|\mathcal{J}^k f\|_{\dot{B}^{-\beta-2k,\infty}_{\infty}}.$$

With these two inequalities at hand, we apply them in (35) and one has

$$\begin{aligned}|\mathcal{J}^{\frac{s_1}{2}} f(x)| &\leq \frac{C}{\Gamma(k-s_1/2)}\left(\int_0^{T} t^{k-\frac{s_1}{2}-1}t^{-k+\frac{s}{2}}\mathcal{M}_B\left(\mathcal{J}^{\frac{s}{2}} f\right)(x)dt + \int_T^{+\infty} t^{k-\frac{s_1}{2}-1}t^{\frac{-\beta-2k}{2}}\|\mathcal{J}^k f\|_{\dot{B}^{-\beta-2k,\infty}_{\infty}}dt\right)\\ &\leq \frac{C}{\Gamma(k-s_1/2)}\left(T^{\frac{s-s_1}{2}}\mathcal{M}_B\left(\mathcal{J}^{\frac{s}{2}} f\right)(x) + T^{\frac{-\beta-s_1}{2}}\|\mathcal{J}^k f\|_{\dot{B}^{-\beta-2k,\infty}_{\infty}}\right).\end{aligned}$$

We fix now the parameter T by the condition

$$T = \left(\frac{\|\mathcal{J}^k f\|_{\dot{B}_\infty^{-\beta-2k,\infty}}}{\mathcal{M}_B\left(\mathcal{J}^{\frac{s}{2}} f\right)(x)} \right)^{\frac{2}{\beta+s}},$$

and we obtain the following inequality

$$|\mathcal{J}^{\frac{s_1}{2}} f(x)| \leq \frac{C}{\Gamma(k - s_1/2)} \mathcal{M}_B\left(\mathcal{J}^{\frac{s}{2}} f\right)^{1-\frac{s-s_1}{\beta+s}}(x) \|\mathcal{J}^k f\|_{\dot{B}_\infty^{-\beta-2k,\infty}}^{\frac{s-s_1}{\beta+s}}.$$

Since $\frac{s-s_1}{\beta+s} = 1 - \theta$ and using again the fact $\|\mathcal{J}^k f\|_{\dot{B}_\infty^{-\beta-2k,\infty}} \simeq \|f\|_{\dot{B}_\infty^{-\beta,\infty}}$ we have

$$|\mathcal{J}^{\frac{s_1}{2}} f(x)| \leq \frac{C}{\Gamma(k - s_1/2)} \mathcal{M}_B\left(\mathcal{J}^{\frac{s}{2}} f\right)^{\theta}(x) \|f\|_{\dot{B}_\infty^{-\beta,\infty}}^{1-\theta}. \tag{36}$$

Once we have obtained this pointwise inequality, we will use the following properties of the nonincreasing rearrangement function.

Lemma 6.2 *If $f, g : \mathbb{G} \longrightarrow \mathbb{R}$ are two measurable functions, we have*

(i) if $|g| \leq |f|$ a.e. then $g^* \leq f^*$,

(ii) if $0 < \theta$, then $(|f|^\theta)^* = (f^*)^\theta$.

For a proof see Proposition 1.4.5 of [22]. Recalling that $\theta = p/q$ and applying these facts to the inequality (36) we obtain

$$\left((\mathcal{J}^{\frac{s_1}{2}} f)^*(t) \right)^q \leq C \left((\mathcal{M}_B\left(\mathcal{J}^{\frac{s}{2}} f\right))^*(t) \right)^p \|f\|_{\dot{B}_\infty^{-\beta,\infty}}^{q-p}. \tag{37}$$

Multiplying the previous inequality by a weight w from the Ariño-Muckenhoupt class B_p and integrating with respect to the variable t we obtain

$$\int_0^{+\infty} \left((\mathcal{J}^{\frac{s_1}{2}} f)^*(t) \right)^q w(t)dt \leq C \int_0^{+\infty} \left((\mathcal{M}_B\left(\mathcal{J}^{\frac{s}{2}} f\right))^*(t) \right)^p w(t)dt \, \|f\|_{\dot{B}_\infty^{-\beta,\infty}}^{q-p},$$

and then, by the definition of classical Lorentz spaces given in Section 4 we have

$$\|\mathcal{J}^{\frac{s_1}{2}} f\|_{\Lambda^q(w)} \leq C \|\mathcal{M}_B(\mathcal{J}^{\frac{s}{2}} f)\|_{\Lambda^p(w)}^\theta \|f\|_{\dot{B}_\infty^{-\beta,\infty}}^{1-\theta}.$$

Now, since the weight w belongs to the class B_p with $1 < p < +\infty$, we have that the Hardy-Littlewood maximal operator is bounded on the space $\Lambda^p(w)$ and we obtain

$$\|\mathcal{M}_B(\mathcal{J}^{\frac{s}{2}} f)\|_{\Lambda^p(w)} \leq \|\mathcal{J}^{\frac{s}{2}} f\|_{\Lambda^p(w)},$$

and finally we have the desired inequality for classical Lorentz spaces:

$$\|\mathcal{J}^{\frac{s_1}{2}} f\|_{\Lambda^q(w)} \leq C \|\mathcal{J}^{\frac{s}{2}} f\|_{\Lambda^p(w)}^\theta \|f\|_{\dot{B}_\infty^{-\beta,\infty}}^{1-\theta}.$$

■

Now we will state in the following corollaries some interesting consequences of this previous theorem.

Corollary 6.1 *Let $w \in B_p$ be a weight and let $f : \mathbb{G} \longrightarrow \mathbb{R}$ be a function such that $f \in \dot{\Lambda}^{s,p,\infty}(w)(\mathbb{G}) \cap \dot{B}_\infty^{-\beta,\infty}(\mathbb{G})$. Then we have the following version of improved Sobolev inequalities of weak type:*

$$\|f\|_{\dot{\Lambda}^{s_1,q,\infty}(w)} \leq C \|f\|_{\dot{\Lambda}^{s,p,\infty}(w)}^\theta \|f\|_{\dot{B}_\infty^{-\beta,\infty}}^{1-\theta},$$

where $1 < p < q < +\infty$, $\theta = p/q$, $s_1 = \theta s - (1-\theta)\beta$ and $-\beta < s_1 < s$.

***Proof*.** We start again with the pointwise inequality (37):

$$\left((\mathcal{J}^{\frac{s_1}{2}}f)^*(t)\right)^q \leq C\left((\mathcal{M}_B\left(\mathcal{J}^{\frac{s}{2}}f\right))^*(t)\right)^p \|f\|^{q-p}_{\dot{B}^{-\beta,\infty}_{\infty}}.$$

Now, we multiply both parts of this inequality by $W(t)$ and we take the supremum in the variable t:

$$\begin{aligned}\|\mathcal{J}^{\frac{s_1}{2}}f\|^q_{\Lambda^{q,\infty}(w)} = \sup_{t>0} W(t)\left((\mathcal{J}^{\frac{s_1}{2}}f)^*(t)\right)^q &\leq C\sup_{t>0}\left\{\left(\mathcal{M}_B(\mathcal{J}^{\frac{s}{2}}f)^*(t)\right)^p W(t)\right\}\|f\|^{q-p}_{\dot{B}^{-\beta,\infty}_{\infty}}\\ &\leq C\|\mathcal{M}_B(\mathcal{J}^{\frac{s}{2}}f)\|^p_{\Lambda^{p,\infty}(w)}\|f\|^{q-p}_{\dot{B}^{-\beta,\infty}_{\infty}},\end{aligned}$$

since it is known (see e.g. [34]) that for $w \in B_p$ the Hardy-Littlewood maximal operator $\mathcal{M}_B$ is bounded on $\Lambda^{p,\infty}(w)$, therefore we obtain that

$$\|\mathcal{J}^{\frac{s_1}{2}}f\|_{\Lambda^{q,\infty}(w)} \leq C\|\mathcal{J}^{\frac{s}{2}}f\|^{\theta}_{\Lambda^{p,\infty}(w)}\|f\|^{1-\theta}_{\dot{B}^{-\beta,\infty}_{\infty}}.$$

■

Now we will study other variations of the previous results by considering a different type of weights. To be more precise, we will study two-weighted inequalities and in what follows, for v and w two weigths and for $t > 0$, we will denote by $V(t)$ and $W(t)$ the quantities $V(t) = \displaystyle\int_0^t v(s)ds$ and $W(t) = \displaystyle\int_0^t w(s)ds$.

Our first two-weighted improved Lorentz-Sobolev inequality is given in the following corollary.

Corollary 6.2 *Let $1 < p < q < +\infty$ and let (v, w) be a pair of positive weights satisfying the following properties*

$$\sup_{t>0}\frac{W(t)^{1/p}}{V(t)^{1/p}} < +\infty \quad \text{and} \quad \sup_{t>0}\left(\int_t^{+\infty}\frac{w(s)}{s^p}ds\right)^{1/p}\left(\int_0^t\frac{v(s)s^{p'}}{V(s)^{p'}}ds\right)^{1/p'} < +\infty.$$

If $f : \mathbb{G} \longrightarrow \mathbb{R}$ is a function such that $f \in \dot{\Lambda}^{s,p}(v) \cap \dot{B}^{-\beta,\infty}_{\infty}$ with $s > 0$, then we have a two-weighted version of improved Sobolev inequalities

$$\|f\|_{\dot{\Lambda}^{s_1,q}(w)} \leq C\|f\|^{\theta}_{\dot{\Lambda}^{s,p}(v)}\|f\|^{1-\theta}_{\dot{B}^{-\beta,\infty}_{\infty}},$$

where $1 < p < q < +\infty$, $\theta = p/q$, $s_1 = \theta s - (1-\theta)\beta$ and $-\beta < s_1 < s$.

This inequality is interesting since it is possible, under some hypotheses, to consider different weights in the left-hand side and in the right-hand side of the inequality.

***Proof*.** Using the pointwise inequality (37) and the fact that the Hardy-Littlewood maximal operator

$$\mathcal{M}_B : \Lambda^p(v) \longrightarrow \Lambda^p(w)$$

is bounded for such weights (see [38] for details) we obtain the desired inequality. ■

If we are allowed to change the weights that define the Lorentz spaces in the previous inequalities, it is then also possible to change, with specific conditions on the weights, the parameters of these spaces. In the following corollary we gather some results where we consider different Lorentz spaces in the right-hand side of the inequality. Indeed, the first point is a generalization of the previous corollary and we will consider in the right-hand side Lorentz-Sobolev spaces of type $\dot{\Lambda}^{s,q_0}(v)$ instead of $\dot{\Lambda}^{s,p}(v)$ where $1 < q_0 \leq p < +\infty$. The second point allows us to study the case when $1 < p < q_0 < +\infty$ and finally, the third point treats the case when $0 < q_0 < 1$.

Corollary 6.3 *Let $0 < q_0 < +\infty$, $s > 0$, let $f : \mathbb{G} \longrightarrow \mathbb{R}$ be a measurable function and let (v, w) be a pair of weights.*

1) If $1 < q_0 \leq p < +\infty$ and if (v, w) are satisfying the following conditions

$$\sup_{t>0}\frac{W(t)^{1/p}}{V(t)^{1/q_0}} < +\infty \tag{38}$$

and

$$\sup_{t>0}\left(\int_0^t \frac{w(s)}{s^p}ds\right)^{1/p}\left(\int_0^t \frac{v(s)s^{q_0'}}{V(s)^{q_0'}}\right)<+\infty, \tag{39}$$

then, if $f\in\dot{\Lambda}^{s,q_0}(v)(\mathbb{G})\cap\dot{B}_\infty^{-\beta,\infty}(\mathbb{G})$*, we have the following inequality*

$$\|f\|_{\dot{\Lambda}^{s_1,q}(w)}\leq C\|f\|^{\theta}_{\dot{\Lambda}^{s,q_0}(v)}\|f\|^{1-\theta}_{\dot{B}_\infty^{-\beta,\infty}},$$

where $1<p<q<+\infty$*,* $\theta=p/q$*,* $s_1=\theta s-(1-\theta)\beta$ *and* $-\beta<s_1<s$*.*

2) If $1<p<q_0<+\infty$ *and* (v,w) *are satisfying*

$$\left(\int_0^{+\infty}\left(\frac{W(s)}{V(s)}\right)^{r/q_0}w(s)ds\right)^{1/r}<+\infty$$

and

$$\left(\int_0^{+\infty}\left[\left(\int_s^{+\infty}\frac{w(t)}{t^p}dt\right)^{1/p}\left(\int_0^t\frac{v(t)t^{q_0{'}}}{V(t)^{q_0{'}}}dt\right)^{1/p'}\right]^r\frac{v(s)s^{q_0{'}}}{V(s)^{q_0{'}}}ds\right)^{1/r}<+\infty,$$

where r *is given by* $\frac{1}{r}=\frac{1}{p}-\frac{1}{q}$*. Then, if* $f\in\dot{\Lambda}^{s,q_0}(v)(\mathbb{G})\cap\dot{B}_\infty^{-\beta,\infty}(\mathbb{G})$*, we have*

$$\|f\|_{\dot{\Lambda}^{s_1,q}(w)}\leq C\|f\|^{\theta}_{\dot{\Lambda}^{s,q_0}(v)}\|f\|^{1-\theta}_{\dot{B}_\infty^{-\beta,\infty}},$$

where $1<p<q<+\infty$*,* $\theta=p/q$*,* $s_1=\theta s-(1-\theta)\beta$ *and* $-\beta<s_1<s$*.*

3) If $0<q_0<1$ *and* $1<p<+\infty$ *and if* (v,w) *are satisfying* (38) *and*

$$\sup_{t>0}\frac{t}{V(t)^{1/q_0}}\left(\int_t^{+\infty}\frac{w(s)}{s^p}ds\right)^{1/p}<+\infty,$$

then, assuming that $f\in\dot{\Lambda}^{s,q_0}(v)(\mathbb{G})\cap\dot{B}_\infty^{-\beta,\infty}(\mathbb{G})$*, we obtain*

$$\|f\|_{\dot{\Lambda}^{s_1,q}(w)}\leq C\|f\|^{\theta}_{\dot{\Lambda}^{s,q_0}(v)}\|f\|^{1-\theta}_{\dot{B}_\infty^{-\beta,\infty}},$$

where $1<p<q<+\infty$*,* $\theta=p/q$*,* $s_1=\theta s-(1-\theta)\beta$ *and* $-\beta<s_1<s$*.*

Proof. From the pointwise inequality (37) we obtain that

$$\|\mathcal{J}^{\frac{s_1}{2}}f\|_{\Lambda^q(w)}\leq C\|\mathcal{M}_B(\mathcal{J}^{\frac{s}{2}}f)\|^{\theta}_{\Lambda^p(w)}\|f\|^{1-\theta}_{\dot{B}_\infty^{-\beta,\infty}}.$$

Now, under all these hypotheses on the weights v and w, we have that the Hardy-Littlewood maximal operator $\mathcal{M}_B:\Lambda^{q_0}(v)\longrightarrow\Lambda^p(w)$ is bounded (see [38] and [7]) and then we obtain

$$\|f\|_{\dot{\Lambda}^{s_1,q}(w)}\leq C\|f\|^{\theta}_{\dot{\Lambda}^{s,q_0}(v)}\|f\|^{1-\theta}_{\dot{B}_\infty^{-\beta,\infty}}.$$

■

We have also the following two-weighted version of improved Sobolev inequalities of weak type:

Corollary 6.4 *Let* $1<p<+\infty$*,* $0<q_0<+\infty$*. Let* (v,w) *be a pair of weights such that*

$$\sup_{t>0}\frac{W(t)^{1/p}}{t}\int_0^t V^{-1/q_0}(s)ds<+\infty, \tag{40}$$

and let $f:\mathbb{G}\longrightarrow\mathbb{R}$ *be a function such that* $f\in\dot{\Lambda}^{s,q_0,\infty}(v)(\mathbb{G})\cap\dot{B}_\infty^{-\beta,\infty}(\mathbb{G})$*. Then we have the following inequality*

$$\|f\|_{\dot{\Lambda}^{s_1,q,\infty}(w)}\leq C\|f\|^{\theta}_{\dot{\Lambda}^{s,q_0,\infty}(v)}\|f\|^{1-\theta}_{\dot{B}_\infty^{-\beta,\infty}},$$

where $0<q_0<+\infty$*,* $1<p<q<+\infty$*,* $\theta=p/q$*,* $s_1=\theta s-(1-\theta)\beta$ *and* $-\beta<s_1<s$*.*

***Proof*.** It is enough to follow the same lines of the Corollary 6.1 to obtain

$$\|\mathcal{J}^{\frac{s_1}{2}}f\|^q_{\Lambda^{q,\infty}(w)} \leq C\|\mathcal{M}_B(\mathcal{J}^{\frac{s}{2}}f)\|^p_{\Lambda^{p,\infty}(w)}\|f\|^{q-p}_{\dot{B}^{-\beta,\infty}_{\infty}},$$

since the pair of weights (v, w) satisfies the condition (40) it implies that the operator $\mathcal{M}_B : \Lambda^{q_0,\infty}(v) \longrightarrow \Lambda^{p,\infty}(w)$ is bounded (see [34]) and we obtain

$$\|\mathcal{J}^{\frac{s_1}{2}}f\|^q_{\Lambda^{q,\infty}(w)} \leq C\|\mathcal{J}^{\frac{s}{2}}f\|^p_{\Lambda^{q,\infty}(v)}\|f\|^{q-p}_{\dot{B}^{-\beta,\infty}_{\infty}},$$

which is the desired inequality. ■

7 Generalizations

In this section we give some generalizations of Theorems 1 and 2 and we prove Theorem 3. These generalizations are made possible since the techniques developed in our proofs are based on general harmonic analysis arguments and since many of the tools used in this article are available in other frameworks. Indeed, the spectral theory associated to the Laplace operator, the boundedness of the Hardy-Littlewood maximal operator and the use of appropiate weights in order to define well suited functional spaces are intensively studied and many interesting properties were generalized to different settings.

7.1 A_p Weighted Inequalities

In this section we consider weights belonging to the A_p class with $1 \leq p < +\infty$ and we will study a weighted version of Theorem 1. B. Muckenhoupt introduced in [29] the A_p class of weights which are also known as Muckenhoupt weights. For the sake of simplicity, we present the tools and the framework in the general setting of stratified Lie groups.

Let us recall first that a weight ω (a locally integrable function on $\mathbb{G}$ with values in $]0, +\infty[$) belongs to the A_1 class if

$$\mathcal{M}_B\,\omega(x) \leq C\,\omega(x) \qquad \text{for all } x \in \mathbb{G},$$

where $\mathcal{M}_B$ is the Hardy-Littlewood maximal function given in (13).

For $1 < p < +\infty$ we say that $\omega \in A_p$ if it satisfies the condition

$$\sup_B\left(\frac{1}{|B|}\int_B \omega(x)dx\right)\left(\frac{1}{|B|}\int_B \omega(x)^{-\frac{1}{p-1}}dx\right)^{p-1} < +\infty, \quad \text{where } B \text{ is an open ball.}$$

It is known that if $1 \leq p < q < +\infty$ we have the inclusion $A_p \subset A_q$. For general properties of A_p weights and more details see [18] and [22]. We define, for $1 \leq p < +\infty$, the weighted Lebesgue spaces by the norm

$$\|f\|_{L^p(\omega)} = \left(\int_{\mathbb{G}} |f(x)|^p\omega(x)dx\right)^{1/p}, \qquad \text{with } \omega \in A_p. \tag{41}$$

Let us notice that we also have a characterization in terms of the distribution function that is

$$\|f\|^p_{L^p(\omega)} = p\int_0^{+\infty} \alpha^{p-1}\omega(\{x \in \mathbb{G} : |f(x)| > \alpha\})d\alpha.$$

We just point out here that one of the main features of A_p weights is related to the boundedness of Hardy-Littlewood maximal function:

$$\int_{\mathbb{G}} \mathcal{M}_Bf(x)^p\omega(x)dx \leq C\int_{\mathbb{G}} |f(x)|^p\omega(x)dx, \qquad f \in L^p(\omega)(\mathbb{G}) \quad \text{with } 1 < p < +\infty.$$

Weighted weak-L^p spaces are given for $1 < p < +\infty$ by

$$\|f\|_{L^{p,\infty}(\omega)} = \sup_{\sigma>0}\{\sigma\,\omega(\{x \in \mathbb{G} : |f(x)| > \sigma\})^{1/p}\}, \qquad \text{with } \omega \in A_p.$$

Once we have defined the weighted $L^p(\omega)$ spaces with the expression (41), we can construct weighted Sobolev spaces in the following manner, for $\omega \in A_p$ with $1 < p < +\infty$, we write:

$$\|f\|_{\dot{W}^{s,p}(\omega)} = \|\mathcal{J}^{s/2} f\|_{L^p(\omega)},$$

and when $p = s = 1$ we have

$$\|f\|_{\dot{W}^{1,1}(\omega)} = \|\nabla f\|_{L^1(\omega)}.$$

With all these definitions and preliminaries, and using the arguments developed in [11] it is possible to adapt the proof of Theorem 1 in the following way.

Theorem 6 *Let ω be a weight in the Muckenhoupt class A_1, if f is a function such that $\nabla f \in L^1(\omega)(\mathbb{G})$ and $f \in \dot{B}^{-\beta,\infty}_{\infty}(\mathbb{G})$, then we have the inequality*

$$\|f\|_{\dot{W}^{s,q}(\omega)} \leq C \|\nabla f\|^{\theta}_{L^1(\omega)} \|f\|^{1-\theta}_{\dot{B}^{-\beta,\infty}_{\infty}},$$

where $1 < q < +\infty$, $0 \leq s < 1/q$, $\beta = \frac{1-sq}{q-1}$ and $\theta = \frac{1}{q}$.

7.2 Morrey spaces

We prove now Theorem 3 in the setting of stratified Lie groups. Morrey spaces were studied in this framework by many authors, see for example the articles [2], [31] and the references there in.

As said in the introduction, once we have at our disposal the fact that the Hardy-Littlewood maximal operator is bounded in the convenient functional framework, it is possible to improve Sobolev inequalities in the following way. The starting point of our proof is the pointwise inequality (36):

$$|\mathcal{J}^{\frac{s_1}{2}} f(x)| \leq \frac{C}{\Gamma(k - s_1/2)} \mathcal{M}_B \left(\mathcal{J}^{\frac{s}{2}} f\right)^{\theta}(x) \|f\|^{1-\theta}_{\dot{B}^{-\beta,\infty}_{\infty}}.$$

Since $\theta = p/q$ we have for $r > 0$ and for $0 \leq a < N$ the inequalities

$$\begin{aligned} \frac{1}{r^a} \int_{B(x_0,r)} |\mathcal{J}^{\frac{s_1}{2}} f(x)|^q dx &\leq C \left(\frac{1}{r^a} \int_{B(x_0,r)} \mathcal{M}_B \left(\mathcal{J}^{\frac{s}{2}} f\right)^p (x) dx \right) \|f\|^{q(1-\theta)}_{\dot{B}^{-\beta,\infty}_{\infty}} \\ \left(\frac{1}{r^a} \int_{B(x_0,r)} |\mathcal{J}^{\frac{s_1}{2}} f(x)|^q dx \right)^{1/q} &\leq C \left(\frac{1}{r^a} \int_{B(x_0,r)} \mathcal{M}_B \left(\mathcal{J}^{\frac{s}{2}} f\right)^p (x) dx \right)^{1/q} \|f\|^{(1-\theta)}_{\dot{B}^{-\beta,\infty}_{\infty}}, \end{aligned}$$

from which we derive the estimate

$$\|\mathcal{J}^{\frac{s_1}{2}} f\|_{\mathcal{M}^{q,a}} \leq C \|\mathcal{M}_B \left(\mathcal{J}^{\frac{s}{2}} f\right)\|^{\theta}_{\mathcal{M}^{p,a}} \|f\|^{(1-\theta)}_{\dot{B}^{-\beta,\infty}_{\infty}}.$$

In order to conclude, we use the fact that the Hardy-Littlewood maximal operator is bounded in Morrey spaces and we obtain

$$\|\mathcal{J}^{\frac{s_1}{2}} f\|_{\mathcal{M}^{q,a}} \leq C \|\mathcal{J}^{\frac{s}{2}} f\|^{\theta}_{\mathcal{M}^{p,a}} \|f\|^{(1-\theta)}_{\dot{B}^{-\beta,\infty}_{\infty}},$$

which is the desired inequality stated in Theorem 3.

Remark 7.1 *The boundedness of the Hardy-Littlewood maximal operator was studied for generalized Morrey spaces in [2], [30] and [33]. As long as this boundedness property is satisfied it should be possible to generalize Theorem 3. Indeed, from the pointwise inequality (36) it should be easy (taking into account the necessary precautions) to reconstruct the corresponding norms in order to obtain an improved Sobolev-like inequality.*

7.3 Nilpotent Lie groups

We consider now a more general framework than the one given by stratified Lie groups. Indeed, going one step further in the process of generalization, it is possible to consider nilpotent Lie groups since all the tools used in the proof of Theorems 1 and 2 are available in these settings.

We recall for the sake of completness this framework. Let $\mathbb{G}$ be a connected unimodular Lie group endowed with its Haar measure dx. Denote by $\mathfrak{g}$ the Lie algebra of $\mathbb{G}$ and consider a family (that will be fixed from now on) of left-invariant vector fields on $\mathbb{G}$

$$\mathbf{X} = \{X_1, ..., X_k\},$$

satisfying the *Hörmander condition*[1]. We endow the group $\mathbb{G}$ with a metric structure by considering the Carnot-Carathéodory metric associated with $\mathbf{X}$. See [39] for details. We will denote $\|x\|$ the distance between the origin e and x and $\|y^{-1} \cdot x\|$ the distance between x and y. For $r > 0$ and $x \in \mathbb{G}$, denote by $B(x, r)$ the open ball with respect to the Carnot-Carathéodory metric centered in x and of radius r, and by $V(r) = \displaystyle\int_{B(x,r)} dx$ the Haar measure of any ball of radius r. When $0 < r < 1$, there exists $d \in \mathbb{N}^*$, c_l and $C_l > 0$ such that, for all $0 < r < 1$ we have

$$c_l r^d \le V(r) \le C_l r^d.$$

The integer d is the *local* dimension of $(\mathbb{G}, \mathbf{X})$. When $r \ge 1$, two situations may occur, independently of the choice of the family $\mathbf{X}$: either $\mathbb{G}$ has polynomial volume growth and there exist $D \in \mathbb{N}^*$, c_∞ and $C_\infty > 0$ such that, for all $r \ge 1$ we have

$$c_\infty r^D \le V(r) \le C_\infty r^D,$$

or $\mathbb{G}$ has exponential volume growth, which means that there exist $c_e, C_e, \alpha, \beta > 0$ such that, for all $r \ge 1$ we have

$$c_e e^{\alpha r} \le V(r) \le C_e e^{\beta r}.$$

When $\mathbb{G}$ has polynomial volume growth, the integer D is called the *dimension at infinity* of $\mathbb{G}$. Recall that nilpotent groups have polynomial volume growth and that a *strict* subclass of the nilpotent groups consists of stratified Lie groups where $d = D$.

Once we have fixed the family $\mathbf{X}$, we define the gradient on $\mathbb{G}$ by $\nabla = (X_1, ..., X_k)$ and we consider a Laplacian $\mathcal{J}$ on $\mathbb{G}$ defined in the same way as in (11)

$$\mathcal{J} = -\sum_{j=1}^{k} X_j^2,$$

which is a positive self-adjoint, hypo-elliptic operator since $\mathbf{X}$ satisfies the Hörmander's condition, see [39]. Its associated heat operator on $]0, +\infty[\times\mathbb{G}$ is given by $\partial_t + \mathcal{J}$ and we will denote by $(H_t)_{t>0}$ the semi-group obtained from the Laplacian $\mathcal{J}$. It is worth noting that many of the properties given in Theorem 4 remain true for the heat semi-group H_t in this general setting. For more details concerning nilpotent Lie groups see the books [39], [18], [36] and the articles [19], [32], [12] and the references there in.

Fractional powers of the Laplacian can be defined in a completely similar way using the expression (12) or using the spectral theory associated to this Laplacian $\mathcal{J}$ just as in Section 3. It is then possible to define all the functional spaces given in Section 4 in the framework of nilpotent Lie groups. Moreover, the Proposition 3.1 is completely available in this general setting.

With all these preliminaries, we see that we have at our disposal all the ingredients needed in order to perform the computations done in Sections 5 and 6, and thus Theorem 1 and Theorem 2 can be generalized to the setting of nilpotent Lie groups.

Acknowledgments. A part of this work was performed while the second and third authors visited the University of Evry Val d'Essonne. We express our gratitude to the Laboratoire de Mathématiques et Modélisation d'Evry (LaMME) of the University of Evry Val d'Essonne for the hospitality and excellent conditions. The second named author was partially supported by POSDRU/159/1.5/S/137750.

[1] which means that the Lie algebra generated by the family $\mathbf{X}$ is $\mathfrak{g}$.

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