

# SHARP GLOBAL WELL-POSEDNESS FOR A HIGHER ORDER SCHRÖDINGER EQUATION

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ABSTRACT. Using the theory of almost conserved energies and the “I-method” developed by Colliander, Keel, Staffilani, Takaoka and Tao, we prove that the initial value problem for a higher order Schrödinger equation is globally well-posed in Sobolev spaces of order  $s > 1/4$ . This result is sharp.

## 1. INTRODUCTION

In this paper we will describe a sharp result of global well-posedness for solutions of the initial value problem (IVP)

$$\begin{cases} \partial_t u + ia \partial_x^2 u + b \partial_x^3 u + ic |u|^2 u + d |u|^2 \partial_x u + e u^2 \partial_x \bar{u} = 0, & x, t \in \mathbb{R}, \\ u(x, 0) = \varphi(x), \end{cases} \quad (1.1)$$

where  $u$  is a complex valued function and  $a, b, c, d$  and  $e$  are real parameters with  $be \neq 0$ .

This model was proposed by Hasegawa and Kodama in [17, 21] to describe the nonlinear propagation of pulses in optical fibers. In literature, this model is called as a higher order nonlinear Schrödinger equation or also Airy-Schrödinger equation.

We consider the following gauge transformation

$$v(x, t) = \exp\left(i\lambda x + i(a\lambda^2 - 2b\lambda^3)t\right) u(x + (2a\lambda - 3b\lambda^2)t, t), \quad (1.2)$$

then,  $u$  solves (1.1) if and only if  $v$  satisfies the IVP

$$\begin{cases} \partial_t v + i(a - 3\lambda b) \partial_x^2 v + b \partial_x^3 v + i(c - \lambda(d - e)) |v|^2 v + d |v|^2 \partial_x v + e v^2 \partial_x \bar{v} = 0, \\ v(x, 0) = \exp(i\lambda x) u(x, 0). \end{cases} \quad (1.3)$$

Thus, if we take  $\lambda = a/3b$  in (1.2) and  $c = (d - e)a/3b$ , then the function

$$v(x, t) = \exp\left(i\frac{a}{3b}x + i\frac{a^3}{27b^2}t\right) u\left(x + \frac{a^2}{3b}t, t\right), \quad (1.4)$$

satisfies the complex modified Korteweg-de Vries type equation

$$\begin{cases} \partial_t v + b \partial_x^3 v + d |v|^2 \partial_x v + e v^2 \partial_x \bar{v} = 0, \\ v(x, 0) = \exp(iax/3b) u(x, 0). \end{cases} \quad (1.5)$$

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Was shown in [22] that the flow associated to the IVP (1.1) leaves the following quantity

$$I_1(u) = \int_{\mathbb{R}} |u|^2(x, t) dx, \quad (1.6)$$

conserved in time. Also, when  $be \neq 0$  we have the following conserved quantity

$$I_2(u) = c_1 \int_{\mathbb{R}} |\partial_x u|^2(x, t) dx + c_2 \int_{\mathbb{R}} |u|^4(x, t) dx + c_3 \operatorname{Im} \int_{\mathbb{R}} u(x, t) \partial_x \overline{u(x, t)} dx, \quad (1.7)$$

where  $c_1 = 3be$ ,  $c_2 = -e(e + d)/2$  and  $c_3 = (3bc - a(e + d))$ . We may suppose  $c_3 = 0$ . In fact, when  $c_3 \neq 0$  we can take in the gauge transformation (1.2)

$$\lambda = -\frac{c_3}{6be}.$$

Then,  $u$  solves (1.1) if and only if  $v$  satisfies (1.3) and in this new IVP we have the constant  $c_3 = 0$ .

We say that the IVP (1.1) is locally well-posed in  $X$  (Banach space) if the solution uniquely exists in certain time interval  $[-T, T]$  (unique existence), the solution describes a continuous curve in  $X$  in the interval  $[-T, T]$  whenever initial data belongs to  $X$  (persistence), and the solution varies continuously depending upon the initial data (continuous dependence) i.e. continuity of application  $u_0 \mapsto u(t)$  from  $X$  to  $\mathcal{C}([-T, T]; X)$ . We say that the IVP (1.1) is globally well-posed in  $X$  if the same properties hold for all time  $T > 0$ . If some hypotheses in the definition of local well-posed fail, we say that the IVP is ill-posed.

Particular cases of (1.1) are the following:

- Cubic nonlinear Schrödinger equation (NLS), ( $a = \mp 1$ ,  $b = 0$ ,  $c = -1$ ,  $d = e = 0$ ).

$$iu_t \pm u_{xx} + |u|^2 u = 0, \quad x, t \in \mathbb{R}. \quad (1.8)$$

The best known local result for the IVP associated to (1.8) is in  $H^s(\mathbb{R})$ ,  $s \geq 0$ , obtained by Tsutsumi [31]. Since the  $L^2$  norm is preserved in (1.8), one has that (1.8) is globally well-posed in  $H^s(\mathbb{R})$ ,  $s \geq 0$ .

- Nonlinear Schrödinger equation with derivative ( $a = -1$ ,  $b = 0$ ,  $c = 0$ ,  $d = 2e$ ).

$$iu_t + u_{xx} + i\lambda(|u|^2 u)_x = 0, \quad x, t \in \mathbb{R}. \quad (1.9)$$

The best known local result for the IVP associated to (1.9) is in  $H^s(\mathbb{R})$ ,  $s \geq 1/2$ , obtained by Takaoka [30]. Colliander et al. [10] they proved that (1.9) is globally well-posed in  $H^s(\mathbb{R})$ ,  $s > 1/2$ .

- Complex modified Korteweg-de Vries (mKdV) equation ( $a = 0$ ,  $b = 1$ ,  $c = 0$ ,  $d = 1$ ,  $e = 0$ ).

$$u_t + u_{xxx} + |u|^2 u_x = 0, \quad x, t \in \mathbb{R}. \quad (1.10)$$

If  $u$  is real, (1.10) is the usual mKdV equation. Kenig et al. [19] proved that the IVP associated to it is locally well-posed in  $H^s(\mathbb{R})$ ,  $s \geq 1/4$  and Colliander et al. [11], proved that (1.10) is globally well-posed in  $H^s(\mathbb{R})$ ,  $s > 1/4$ .

• When  $a \neq 0$  is real and  $b = 0$ , we obtain a particular case of the well-known mixed nonlinear Schrödinger equation

$$u_t = iau_{xx} + \lambda(|u|^2)_x u + g(u), \quad x, t \in \mathbb{R}, \quad (1.11)$$

where  $g$  satisfies some appropriate conditions and  $\lambda \in \mathbb{R}$  is a constant. Ozawa and Tsutsumi in [24] proved that for any  $\rho > 0$ , there is a positive constant  $T(\rho)$  depending only on  $\rho$  and  $g$ , such that the IVP (1.11) is locally well-posed in  $H^{1/2}(\mathbb{R})$ , whenever the initial data satisfies

$$\|u_0\|_{H^{1/2}} \leq \rho.$$

There are other dispersive models similar to (1.1), see for instance [1, 8, 25, 26, 28] and the references therein.

Regarding the IVP (1.1), Laurey in [22] showed that the IVP is locally well-posed in  $H^s(\mathbb{R})$  with  $s > 3/4$ , and using the quantities (1.6) and (1.7) she proved the global well-posedness in  $H^s(\mathbb{R})$  with  $s \geq 1$ . In [27] Staffilani established the local well-posedness in  $H^s(\mathbb{R})$  with  $s \geq 1/4$ , for the IVP associated to (1.1), improving Laurey's result.

In the IVP (1.1), when  $a, b$  are real functions of  $t$ , in [4, 6] was prove the local well-posedness in  $H^s(\mathbb{R})$ ,  $s \geq 1/4$ . Also, in [4, 7] was study the unique continuation property for the solution of (1.1).

**Remark 1.1. 1)** Using (1.4) and the results obtained in [11] we have that the PVI (1.1) is globally well-posed in  $H^s(\mathbb{R})$  with  $s > 1/4$ , for initial data of the form:

$$\exp\{-i\frac{a}{3b}x\}v_0(x), \quad \exp\{-i\frac{a}{3b}x\}(v_0(x) + iv_0(x)),$$

where  $v_0 \in H^s$ ,  $s > 1/4$ ,  $v_0 \in \mathbb{R}$ . Therefore it suggests us to improve the result and obtain the global existence for the general case in  $H^s(\mathbb{R})$ ,  $s > 1/4$ .

**2)** If  $e = 0$ ,  $bd > 0$  and  $c = (a/3b)d$  in (1.1), then the equation

$$\partial_t u + ia\partial_x^2 u + b\partial_x^3 u + i\frac{a}{3b}d|u|^2 u + d|u|^2 \partial_x u = 0, \quad (1.12)$$

have the following solution with two parameters

$$u_{\eta, N}(x, t) = f_{\eta}(x + \psi(\eta, N)t) \exp i\{Nx + \phi(\eta, N)t\}, \quad (1.13)$$

where  $f_{\eta}(x) = \eta f(\eta x)$ ,  $f(x) = (A \cosh x)^{-1}$ ,  $A = \sqrt{d/(6b)}$ ,  $\psi(\eta, N) = 2aN + 3bN^2 - \eta^2 b$  and  $\phi(\eta, N) = aN^2 + bN^3 - 3\eta^2 bN - a\eta^2$ .

Using the transformation (1.4) we can to obtain other family of solutions for (1.12). In fact, let  $w$  solution of

$$\begin{cases} \partial_t w + \partial_x^3 w + |w|^2 \partial_x w = 0, & x, t \in \mathbb{R}, \\ w(x, 0) = w_0(x) = f_1(x) \exp i\{Nx\} = (\frac{1}{\sqrt{6}} \cosh x)^{-1} \exp i\{Nx\}, \end{cases} \quad (1.14)$$

given by (1.13). If  $w$  is a solution of (1.14), then

$$v(x, t) = \frac{1}{\alpha} w(b^{-1/3}x, t), \quad \alpha = \sqrt{\frac{d}{b^{1/3}}}$$

is a solution of

$$\begin{cases} \partial_t v + b\partial_x^3 v + d|v|^2 \partial_x v = 0, & x, t \in \mathbb{R}, \\ v(x, 0) = v_0(x), \end{cases} \quad (1.15)$$

with initial data  $v_0(x) = (1/\alpha)w(b^{-1/3}x, 0)$  and if  $v$  is a solution of (1.15) then, using the transformation (1.4)

$$u(x, t) = v(x - \frac{a^2}{3b}t, t) \exp i(\frac{2a^3}{27b^2}t - \frac{a}{3b}x)$$

is a solution of (1.12) with initial data  $u_0(x) = v(x, 0) \exp \{-i(a/3b)x\}$ , therefore other solution of (1.12) with two parameters is

$$u_{\eta, N}(x, t) = g_{\eta}(b^{-1/3}x + \psi(\eta, N)t) \exp \{ix(b^{-1/3}N - \frac{a}{3b}) + it\phi(\eta, N)\}, \quad (1.16)$$

where  $g(x) = (\tilde{\alpha} \cosh x)^{-1}$ ,  $\tilde{\alpha} = \alpha/\sqrt{6}$ ,  $\phi(\eta, N) = 2a^3/(27b^2) - 3N\eta^2 + N^3 - Na^2b^{-1/3}/(3b)$ ,  $\psi(\eta, N) = -a^2b^{-1/3}/(3b) - \eta^2 + 3N^2$  and

$$u_{\eta, N}(x, 0) = u_{0\eta, N}(x) = g_{\eta}(b^{-1/3}x) \exp \{ix(b^{-1/3}N - \frac{a}{3b})\}.$$

When  $a = 0$  and  $b = d = 1$  in (1.12), this solution coincide with the solution obtained in [20].

**3)** If  $e \neq 0$  and  $b(d + e) > 0$ , then (1.1) have solutions with one parameter:

$$u_{\eta}(x, t) = g_{\eta}(x + \psi(\eta, w)t) \exp i\{wx + \phi(\eta, w)t\},$$

where  $w = (c - 2aA^2)/(2e)$ ,  $g_{\eta}(x) = \eta g(\eta x)$ ,  $g(x) = (A \cosh x)^{-1}$ ,  $A = \sqrt{(e + d)/(6b)}$ ,  $\psi$  and  $\phi$  as in (1.13).

We have also that if  $u$  is a solution of (1.1) then,  $v = \alpha u$  is a solution of (1.1), where  $\alpha \in \mathbb{C}$ ,  $|\alpha| = 1$ , and if  $d \neq e$  in (1.1) then  $u(x, t) = \exp i\{Cx + Dt + C_0\}$  is a solution of (1.1), where  $D = aC^2 + bC^3$  e  $C = c/(e - d)$ .

Recently there appeared several papers devoted to the global solution of the dispersive type equation, where the framework is based on almost conserved laws and the I-method, see [9, 10, 11, 12, 13]. In this paper we adopt this way in order to obtain our results.

Our aim in this paper is to extend the local solution to a global one. Now, we state our main theorem of global existence:

**Theorem 1.2.** *The IVP (1.1), with  $c = (d - e)a/3b$ , is global well-posedness in  $H^s$ ,  $s > 1/4$ .*

**Notation.** The notation to be used is mostly standard. We will use the space-time Lebesgue  $L_x^p L_T^q$  endowed with the norm

$$\|f\|_{L_x^p L_T^q} = \|\|f\|_{L_T^q}\|_{L_x^p} = \left( \int_{\mathbb{R}} \left( \int_0^T |f(x, t)|^q dt \right)^{p/q} dx \right)^{1/p}.$$

We will use the notation  $\|f\|_{L_x^p L_t^q}$  when the integration in the time variable is on the whole real line. In order to define the  $X_{s, \beta}$  spaces we consider the following IVP

$$\begin{cases} u_t + iau_{xx} + bu_{xxx} = 0, & x, t \in \mathbb{R}, b \neq 0, \\ u(0) = u_0, \end{cases}$$

whose solution is given by  $u(x, t) = U(t)u_0(x)$ , where the unitary group  $U(t)$  is defined as

$$\widehat{U(t)u_0}(\xi) = e^{it(b\xi^3 + a\xi^2)} \widehat{u_0}(\xi).$$

For  $s, \beta \in \mathbb{R}$ ,  $X_{s, \beta}$  denotes the completion of the Schwartz space  $S(\mathbb{R}^2)$  with respect to the norm

$$\begin{aligned} \|u\|_{s, \beta} &\equiv \|u\|_{X_{s, \beta}} \equiv \|U(-t)u\|_{H_{s, \beta}} \equiv \|\langle \tau \rangle^\beta \langle \xi \rangle^s \widehat{U(-t)u}(\xi, \tau)\|_{L_\tau^2 L_\xi^2} \\ &= \|\langle \tau - (b\xi^3 + a\xi^2) \rangle^\beta \langle \xi \rangle^s \widehat{u}(\xi, \tau)\|_{L_\tau^2 L_\xi^2}, \end{aligned}$$

where

$$\widehat{u}(\xi, \tau) \equiv \int_{\mathbb{R}^2} e^{-i(x\xi + t\tau)} u(x, t) dx dt.$$

For any time interval  $[0, \rho]$ , we define the space  $X_{s, b}^\rho$  by the norm

$$\|u\|_{X_{s, b}^\rho} = \inf\{\|U\|_{X_{s, b}} : U|_{[0, \rho] \times \mathbb{R}} = u\}.$$

The notation  $A \lesssim B$  means there exist a constant  $C$  such that  $A \leq C B$ , and  $A \sim B$  means  $A \lesssim B$  and  $B \lesssim A$ . The notations  $\xi_{ij}$  means  $\xi_i + \xi_j$ ,  $\xi_{ijk}$  means  $\xi_i + \xi_j + \xi_k$ , etc. Also we use the notation  $m(\xi_i) := m_i$ ,  $m(\xi_{ij}) := m_{ij}$ , etc.

The notations for multilinear expressions is the same as in [9, 10], we define a spatial n-multiplier to be any function  $M_n(\xi_1, \dots, \xi_n)$  on the hyperplane

$$\Gamma_n := \{(\xi_1, \dots, \xi_n) \in \mathbb{R}^n; \xi_1 + \dots + \xi_n = 0\},$$

which we endow with the dirac measure  $\delta(\xi_1 + \dots + \xi_n)$ . We define the n-linear functional as

$$\Lambda(M_n; f_1, \dots, f_n) := \int_{\Gamma_n} M_n(\xi_1, \dots, \xi_n) \prod_1^n \widehat{f_j}(\xi_j),$$

where  $f_1, \dots, f_n$  are complex functions on  $\mathbb{R}$ . We shall denote

$$\Lambda(M_n; f) := \Lambda(M_n; f, \bar{f}, f, \bar{f}, \dots, f, \bar{f}).$$

For  $1 \leq j \leq n$ ,  $k \geq 1$  we define the elongation  $\mathbf{X}_j^k(M_n)$  of  $M_n$  to be the multiplier of order  $n+k$  given by

$$\mathbf{X}_j^k(M_n)(\xi_1, \dots, \xi_{n+k}) := M_n(\xi_1, \dots, \xi_{j-1}, \xi_j, \dots, \xi_{j+k}, \xi_{j+k+1}, \dots, \xi_{n+k}).$$

## 2. ALMOST CONSERVATIONS LAWS

From (1.1) we have

$$\begin{aligned} \partial_t w + ia \partial_x^2 w + b \partial_x^3 w + ic w \bar{w} w + d (\partial_x w) \bar{w} w + e w (\partial_x \bar{w}) w &= 0, \\ \partial_t \bar{w} - ia \partial_x^2 \bar{w} + b \partial_x^3 \bar{w} - ic \bar{w} w \bar{w} + d (\partial_x \bar{w}) w \bar{w} + e \bar{w} (\partial_x w) \bar{w} &= 0. \end{aligned}$$

Taking Fourier transformation in the above equalities we obtain the following result

**Proposition 2.1.** *Let  $n \geq 2$ , be an even integer, and let  $M_n$  be a multiplier of order  $n$ , then*

$$\begin{aligned} \partial_t \Lambda_n(M_n; w) &= i \Lambda_n(M_n \Upsilon_n^{a,b}; w) - i \Lambda_{n+2} \left( \sum_{j=1}^n \Upsilon_{j,n+2}^{c,e} \mathbf{X}_j^2(M_n); w \right) \\ &\quad - id \Lambda_{n+2} \left( \sum_{j=1}^{n/2} \mathbf{X}_{2j-1}^2(M_n) \xi_{2j-1} + \sum_{j=1}^{n/2} \mathbf{X}_{2j}^2(M_n) \xi_{2j+2}; w \right), \end{aligned} \quad (2.1)$$

where  $\Upsilon_n^{a,b} = \sum_{j=1}^n ((-1)^{j-1} a \xi_j^2 + b \xi_j^3)$  and  $\Upsilon_{j,n+2}^{c,e} = (-1)^{j-1} c + e \xi_{j+1}$ .

We define the first modified energy as

$$E_1 = k_1 \Lambda_2(M_2; w), \quad M_2(\xi_1, \xi_2) = \xi_1 \xi_2 m(\xi_1) m(\xi_2), \quad (2.2)$$

where  $k_1 = 3be$ , and the second modified energy as

$$E_2 = E_1 + \Lambda_4(\delta_4), \quad (2.3)$$

where the 4-multiplier  $\delta_4$  will be choosed after. By (2.1) we get

$$\begin{aligned} \partial_t E_2 &= \partial_t E_1 + \partial_t \Lambda_4(\delta_4) = k_1 \Lambda_2(M_2 \Upsilon_2^{a,b}) - ik_1 \Lambda_4 \left( \sum_{j=1}^2 \Upsilon_{j,4}^{c,e} \mathbf{X}_j^2(M_2) \right) \\ &\quad - idk_1 \Lambda_4(\mathbf{X}_1^2(M_2) \xi_1 + \mathbf{X}_2^2(M_2) \xi_4) + i \Lambda_4(\delta_4 \Upsilon_4^{a,b}) - i \Lambda_6 \left( \sum_{j=1}^4 \Upsilon_{j,6}^{c,e} \mathbf{X}_j^2(\delta_4) \right) \\ &\quad - id \Lambda_6 \left( \sum_{j=1}^2 \mathbf{X}_{2j-1}^2(\delta_4) \xi_{2j-1} + \sum_{j=1}^2 \mathbf{X}_{2j}^2(\delta_4) \xi_{2j+2} \right), \end{aligned} \quad (2.4)$$

it is clear that  $\Lambda_2(M_2 \Upsilon_2^{a,b}) = 0$ . Now if  $\tilde{M}_n$  is an  $n$ - multiplier  $\Lambda_n(\tilde{M}_n)$  is invariant under permutations of the even  $\xi_j$  indices or of the odd  $\xi_j$  indices, therefore for

achieve a cancellation of the 4-linear expression, we choose  $\delta_4$  such that

$$\begin{aligned} \Upsilon_4^{a,b} \delta_4 &= \frac{ck_1}{2} (\xi_1^2 m_1^2 - \xi_2^2 m_2^2 + \xi_3^2 m_3^2 - \xi_4^2 m_4^2) - \frac{ek_1}{2} (\xi_2 \xi_4^2 m_4^2 + \xi_4 \xi_2^2 m_2^2 + \xi_1 \xi_3^2 m_3^2 + \xi_3 \xi_1^2 m_1^2) \\ &\quad - \frac{dk_1}{2} (\xi_1 \xi_4^2 m_4^2 + \xi_4 \xi_1^2 m_1^2 + \xi_3 \xi_2^2 m_2^2 + \xi_2 \xi_3^2 m_3^2), \end{aligned} \quad (2.5)$$

consequently from (2.4) we get

$$\partial_t E_2 = \Lambda_6(\delta_6), \quad (2.6)$$

with

$$\begin{aligned} \delta_6 &= \frac{-ie}{36} \sum_{\substack{\{k,m,o\}=\{1,3,5\} \\ \{l,n,p\}=\{2,4,6\}}} [\xi_l \delta_4(\xi_{klm}, \xi_n, \xi_o, \xi_p) + \xi_m \delta_4(\xi_k, \xi_{lmn}, \xi_o, \xi_p) + \xi_n \delta_4(\xi_k, \xi_l, \xi_{mno}, \xi_p) \\ &\quad + \xi_o \delta_4(\xi_k, \xi_l, \xi_m, \xi_{nop})] - \frac{id}{36} \sum_{\substack{\{k,m,o\}=\{1,3,5\} \\ \{l,n,p\}=\{2,4,6\}}} [\xi_k \delta_4(\xi_{klm}, \xi_n, \xi_o, \xi_p) + \xi_m \delta_4(\xi_k, \xi_l, \xi_{mno}, \xi_p) \\ &\quad + \xi_n \delta_4(\xi_k, \xi_{lmn}, \xi_o, \xi_p) + \xi_p \delta_4(\xi_k, \xi_l, \xi_m, \xi_{nop})]. \end{aligned}$$

**Proposition 2.2.** *If  $m(\xi) = 1$  for all  $\xi$ , then*

$$\partial_t E_2 = 0.$$

*Proof.* From definition of  $E_2$ , we have

$$E_2 = 3be\Lambda_2(\xi_1 \xi_2; w) + \Lambda_4(\delta_4; w), \quad (2.7)$$

where

$$\begin{aligned} \Upsilon_4^{a,b} \delta_4 &= \frac{ck_1}{2} (\xi_1^2 - \xi_2^2 + \xi_3^2 - \xi_4^2) - \frac{ek_1}{2} (\xi_2 \xi_4^2 + \xi_4 \xi_2^2 + \xi_1 \xi_3^2 + \xi_3 \xi_1^2) \\ &\quad - \frac{dk_1}{2} (\xi_1 \xi_4^2 + \xi_4 \xi_1^2 + \xi_3 \xi_2^2 + \xi_2 \xi_3^2). \end{aligned}$$

If  $\xi_1 + \dots + \xi_4 = 0$ , then  $\xi_1^2 - \xi_2^2 + \xi_3^2 - \xi_4^2 = 2\xi_{12}\xi_{14}$  and  $\xi_1^3 + \dots + \xi_4^3 = 3\xi_{12}\xi_{13}\xi_{14}$ , therefore

$$\Upsilon_4^{a,b} = 2a\xi_{12}\xi_{14} + 3b\xi_{12}\xi_{13}\xi_{14} = \xi_{12}\xi_{14}(2a + 3b\xi_{13}). \quad (2.8)$$

On the other hand  $\xi_2 \xi_4^2 + \xi_4 \xi_2^2 + \xi_1 \xi_3^2 + \xi_3 \xi_1^2 = -\xi_{12}\xi_{13}\xi_{14}$  (see Lemma 3.5 in [9] and Remark 3.6 in [10]), similarly  $\xi_1 \xi_4^2 + \xi_4 \xi_1^2 + \xi_3 \xi_2^2 + \xi_2 \xi_3^2 = -\xi_{12}\xi_{13}\xi_{14}$ , hence

$$\begin{aligned} \delta_4 &= \frac{3be}{2} \frac{2c\xi_{12}\xi_{14} + (d+e)\xi_{12}\xi_{13}\xi_{14}}{\xi_{12}\xi_{14}(2a + 3b\xi_{13})} \\ &= \frac{e(d+e)}{2}. \end{aligned}$$

And from (2.7) we get

$$\begin{aligned} E_2 &= 3be\Lambda_2(\xi_1\xi_2; w) + \frac{e(d+e)}{2}\Lambda_4(1; w) \\ &= -3be \int_{\mathbb{R}} |w_x|^2 + \frac{e(d+e)}{2} \int_{\mathbb{R}} |w|^4 \\ &= -I_2(w). \end{aligned}$$

This concludes the proof of the proposition.  $\square$

In the following sections we will consider  $a = c = 0$  in the IVP (1.1) (see (1.4) and (1.5)).

### 3. PRELIMINARY RESULTS

For the estimates on the multipliers we use the following elementary results.

**Lemma 3.1. 1)** (*Double mean value theorem DMVT*)

Let  $f \in C^2(\mathbb{R})$ , and  $\max\{|\eta|, |\lambda|\} \ll \xi$ , then

$$|f(\xi + \eta + \lambda) - f(\xi + \eta) - f(\xi + \lambda) + f(\xi)| \lesssim |f''(\theta)| |\eta| |\lambda|,$$

where  $|\theta| \sim |\xi|$ .

**2)** (*Triple mean value theorem TMVT*)

Let  $f \in C^3(\mathbb{R})$ , and  $\max\{|\eta|, |\lambda|, |\gamma|\} \ll \xi$ , then

$$\begin{aligned} &|f(\xi + \eta + \lambda + \gamma) - f(\xi + \lambda + \eta) - f(\xi + \eta + \gamma) - f(\xi + \lambda + \gamma) + f(\xi + \eta) \\ &\quad + f(\xi + \lambda) + f(\xi + \gamma) - f(\xi)| \lesssim |f'''(\theta)| |\eta| |\lambda| |\gamma|, \end{aligned}$$

where  $|\theta| \sim |\xi|$ .

And for the proof of Proposition 5.2, shall be fundamental the improved Strichartz estimate.

**Lemma 3.2.** Let  $s > 1/4$ ,  $v_1, v_2 \in \mathbf{S}(\mathbb{R} \times \mathbb{R})$  such that  $\text{supp } \widehat{v}_1 \subset \{|\xi| \sim N\}$  and  $\text{supp } \widehat{v}_2 \subset \{|\xi| \ll N\}$ , then

$$\|v_1 v_2\|_{L_x^4 L_t^2} \lesssim \frac{1}{(1-4s)^{1/4}} \frac{1}{N} \|v_2\|_{X_{s,1/2+}^\rho} \|v_1\|_{X_{0,1/2+}^\rho}.$$

*Proof.* As in [9] is sufficient to prove

$$\|v_1 v_2\|_{L_x^4 L_t^2} \lesssim \frac{1}{(1-4s)^{1/4}} \frac{1}{N} \|\phi\|_{H^s} \|\psi\|_{L^2},$$



where  $v_1 = U(t)\psi$  and  $v_2 = U(t)\phi$ . By duality, definition of  $v_1, v_2$ , Fubini theorem and Plancherel identity in the spatial variable we have

$$\begin{aligned} \|v_1 v_2\|_{L_x^4 L_t^2} &\lesssim \sup_{\|F\|_{L_x^{4/3} L_t^2} \leq 1} \int_{\mathbb{R}^2} |\widehat{\phi}(y)\widehat{\psi}(z)\widehat{F}(z+y, z^3+y^3)| dz dy \\ &= \frac{1}{N^2} \sup_{\|F\|_{L_x^{4/3} L_t^2} \leq 1} \int_{\mathbb{R}^2} |\widehat{\phi}(y(s, r))\widehat{\psi}(z(s, r))\widehat{F}(s, r)| dr ds, \end{aligned}$$

where we used the change of variable  $z+y=s$ ,  $z^3+y^3=r$ , which has Jacobian of size  $N^2$ . Now if we applying Hölder inequality and a change of variables back for  $z$  and  $y$ , we obtain

$$\begin{aligned} \|v_1 v_2\|_{L_x^4 L_t^2} &\lesssim \frac{1}{N^2} \sup_{\|F\|_{L_x^{4/3} L_t^2} \leq 1} \|\widehat{\phi}(y(s, r))\widehat{\psi}(z(s, r))\|_{L_s^{4/3} L_r^2} \|\widehat{F}\|_{L_x^4 L_t^2} \\ &\leq \frac{1}{N} \sup_{\|F\|_{L_x^{4/3} L_t^2} \leq 1} \|\psi\|_{L_z^2} \|\widehat{\phi}\|_{L_y^{4/3}} \|\widehat{F}\|_{L_x^4 L_t^2}, \end{aligned}$$

where the Fourier transform of  $F$  is taking only in the space variable. Using Hölder inequality we obtain for  $s > 1/4$

$$\int_{\mathbb{R}} |\widehat{\phi}|^{4/3} \leq \left( \int_{\mathbb{R}} \langle \xi^2 \rangle^s |\widehat{\phi}|^2 \right)^{2/3} \left( \int_{\mathbb{R}} \frac{1}{\langle \xi^2 \rangle^{2s}} \right)^{1/3},$$

therefore

$$\|\widehat{\phi}\|_{L_y^{4/3}} \leq \frac{1}{(1-4s)^{1/4}} \|\phi\|_{H^s},$$

and by Hausdorff-Young inequality and Minkowsky integral inequality, we get

$$\|\widehat{F}\|_{L_x^4 L_t^2} \leq \|\widehat{F}\|_{L_t^2 L_x^4} \leq \|F\|_{L_t^2 L_x^{4/3}} \leq \|F\|_{L_x^{4/3} L_t^2} \leq 1.$$

This completes the proof.  $\square$

We define the Fourier multiplier operator  $I$  with symbol

$$m(\xi) = \begin{cases} 1, & |\xi| < N, \\ \frac{N^{1-s}}{|\xi|^{1-s}}, & |\xi| > 2N. \end{cases} \quad (3.1)$$

We have  $I : H^s \mapsto H^1$ . For the local result we define the Fourier multiplier operator  $L$ , with symbol

$$l(\xi) = m(\xi)\langle \xi \rangle^{1-s} = \begin{cases} \langle \xi \rangle^{1-s}, & |\xi| < N, \\ \langle \xi \rangle^{1-s} \frac{N^{1-s}}{|\xi|^{1-s}}, & |\xi| > 2N. \end{cases}$$

Is obvious that

$$\|Iu\|_{H^1} = \|Lu\|_{H^s}, \quad \|Iu\|_{X_{1,b}} = \|Lu\|_{X_{s,b}}, \quad (3.2)$$

and for  $s \in [0, 1)$  is  $1 \leq l(\xi) \lesssim N^{1-s}$ , therefore

$$\|u\|_{s', b'} \lesssim \|Iu\|_{s'-s+1, b'} \lesssim N^{1-s} \|u\|_{s', b'}, \quad s \in [0, 1),$$

observe that if  $V|_{[0,\rho] \times \mathbb{R}} = Iu$ ,  $V \in X_{s'-s+1, b'}$ , then  $U$  defined by  $\widehat{U} = (1/m)\widehat{V}$ , satisfies

$$\|U\|_{s', b'} \lesssim \|V\|_{s'-s+1, b'},$$

moreover in  $[0, \rho]$  is  $U|_{[0,\rho] \times \mathbb{R}} = u$ , therefore

$$\|u\|_{X_{s', b'}^\rho} \lesssim \|Iu\|_{X_{s'-s+1, b'}^\rho}. \quad (3.3)$$

Also we have

$$l(\xi_1 + \xi_2) \lesssim l(\xi_1) + l(\xi_2). \quad (3.4)$$

In fact, for see this, without lost of generality we can assume  $|\xi_1| \geq |\xi_2|$ , we consider two cases:

i) If  $|\xi_1| \leq N$ , then we have  $|\xi_1 + \xi_2| \leq 2N$ , this implies

$$l(\xi_1 + \xi_2) \sim \langle \xi_1 + \xi_2 \rangle^{1-s} \leq \langle \xi_1 \rangle^{1-s} + \langle \xi_2 \rangle^{1-s} = l(\xi_1) + l(\xi_2).$$

ii) If  $|\xi_1| \geq N$ , then we have  $l(\xi_1) \sim N^{1-s}$ , thus for all  $\xi$ ,  $l(\xi) \lesssim l(\xi_1)$ , in particular  $l(\xi_1 + \xi_2) \lesssim l(\xi_1) \leq l(\xi_1) + l(\xi_2)$ . Note that (3.4) implies  $l(\xi_1 + \xi_2) \lesssim l(\xi_1)l(\xi_2)$ .

In the proof of Theorem 1.2 we will use the following local result.

**Theorem 3.3.** *Let  $s \geq 1/4$ , then the IVP (1.1) is locally well-posed for data  $\varphi$ , with  $I\varphi \in H^1$  where the time of existence satisfies*

$$\delta \sim \|I\varphi\|_{H^1}^{-\theta}, \quad (3.5)$$

with  $\theta > 0$ . Moreover the solution of the IVP (1.1), is such that

$$\|Iu\|_{X_{1, 1/2+}^\delta} \lesssim \|Iu_0\|_{H^1}. \quad (3.6)$$

*Proof.* The Theorem 3.3 is practically done in [29] (see also [32]), in fact, is sufficient to prove

$$\|L(uv\overline{w}_x)\|_{X_{s, -1/2+}} \lesssim \|Lu\|_{X_{s, 1/2+}} \|Lv\|_{X_{s, 1/2+}} \|Lw\|_{X_{s, 1/2+}}, \quad (3.7)$$

$$\|L(u\overline{v}w_x)\|_{X_{s, -1/2+}} \lesssim \|Lu\|_{X_{s, 1/2+}} \|Iv\|_{X_{s, 1/2+}} \|Lw\|_{X_{s, 1/2+}},$$

$$\|L(u\overline{v}w)\|_{X_{s, -1/2+}} \lesssim \|Lu\|_{X_{s, 1/2+}} \|Lv\|_{X_{s, 1/2+}} \|Lw\|_{X_{s, 1/2+}}, \quad (3.8)$$

in order to prove the first inequality we make the following decomposition

$$\begin{aligned} l(\xi) \widehat{uv\overline{w}_x}(\xi, \tau) = & l(\xi) \int_{|\xi_1| > 2N} \zeta + l(\xi) \int_{\substack{|\xi_1| \leq 2N \\ |\xi_2| > 2N}} \zeta + l(\xi) \int_{\substack{|\xi_1| \leq 2N \\ |\xi_2| \leq 2N \\ |\xi - \xi_1 - \xi_2| > 2N}} \zeta \\ & + l(\xi) \int_{\substack{|\xi_1| \leq 2N \\ |\xi_2| \leq 2N \\ |\xi - \xi_1 - \xi_2| \leq 2N}} \zeta, \end{aligned}$$

where  $\zeta := \widehat{u}(\xi - \xi_1 - \xi_2, \tau - \tau_1 - \tau_2)\widehat{v}(\xi_1, \tau_1)\widehat{w}_x(\xi_2, \tau_2)$ , thus

$$|l(\xi)\widehat{uv\overline{w}}_x(\xi, \tau)| \lesssim |\widehat{uv_1\overline{w}}_x| + |\widehat{uv_2\overline{v_3}}_x| + |\widehat{v_2\overline{v_4}v_5}| + \langle \xi \rangle^{1-s} |\widehat{v_2\overline{v_4}v_6}|,$$

with

$$\begin{aligned} \widehat{v}_1(\xi, \tau) &= \chi_{|\xi| > 2N} \widehat{v}(\xi, \tau) l(\xi), & \widehat{v}_2(\xi, \tau) &= \chi_{|\xi| \leq 2N} \widehat{v}(\xi, \tau), \\ \widehat{v_3}_x(\xi, \tau) &= \chi_{|\xi| > 2N} \widehat{w}_x(\xi, \tau) l(\xi), & \widehat{v_4}_x(\xi, \tau) &= \chi_{|\xi| \leq 2N} \widehat{w}_x(\xi, \tau), \\ \widehat{v}_5(\xi, \tau) &= \chi_{|\xi| > 2N} \widehat{u}(\xi, \tau) l(\xi), & \widehat{v}_6(\xi, \tau) &= \chi_{|\xi| \leq 2N} \widehat{u}(\xi, \tau), \end{aligned}$$

and applying Proposition 2.7 in [29] (see also Theorem 2.1 in [32]) we obtain (3.7). For the other inequalities we make an analogous decomposition.  $\square$

We also have a result of local well-posed with the interval of existence as in (3.5), without the use of the theory of the spaces  $X_{s,b}$ .

In fact, let

$$\begin{aligned} |||u|||_{\Delta T, s} &= \|\partial_x u\|_{L_x^\infty L_{\Delta T}^2} + \|D_x^s \partial_x u\|_{L_x^\infty L_{\Delta T}^2} + \|D_x^{s-1/4} \partial_x u\|_{L_x^{20} L_{\Delta T}^{5/2}} + \|u\|_{L_x^5 L_{\Delta T}^{10}} \\ &\quad + \|D_x^s u\|_{L_x^5 L_{\Delta T}^{10}} + \|u\|_{L_x^4 L_{\Delta T}^\infty} + \|u\|_{L_x^8 L_{\Delta T}^8} + \|D_x^s u\|_{L_x^8 L_{\Delta T}^8}. \end{aligned}$$

**Theorem 3.4.** *Let  $s \geq 1/4$ , and  $a, b \in \mathbb{R}, b \neq 0, c, d, e \in \mathbb{C}$ , then the IVP (1.1) is locally well-posed for data  $\varphi$ , with  $I\varphi \in H^1$ . Moreover the solution is such that*

$$|||u|||_{\delta, s} \lesssim \|I\varphi\|_{H^1},$$

where  $\delta$  satisfies (3.5) with  $\theta = 4$ .

*Proof.* The theorem follows from the proof in [27] if we prove

$$|||U(t)u_0|||_{\delta, s} \lesssim \|Iu_0\|_{H^1} = \|Lu_0\|_{H^s},$$

we consider the first term in the definition of  $|||\cdot|||_T$ , we will prove

$$\|\partial_x U(t)u_0\|_{L_x^\infty L_\delta^2} \lesssim \|Lu_0\|_{L^2}. \quad (3.9)$$

The inequality (3.9) is equivalent with

$$\|L^{-1} \partial_x U(t)u_0\|_{L_x^\infty L_\delta^2} \lesssim \|u_0\|_{L^2},$$

where the Fourier multiplier operator  $L^{-1}$  have symbol  $1/l(\xi) \leq 1$ , it is easy to see that

$$\|L^{-1} \partial_x U(t)u_0\|_{L_x^\infty L_\delta^2} \leq \|\partial_x U(t)L^{-1}u_0\|_{L_x^\infty L_\delta^2} \leq \|L^{-1}u_0\|_{L^2} \leq \|u_0\|_{L^2}. \quad (3.10)$$

We proceed similarly with the others terms.  $\square$

**Lemma 3.5.** *For any  $s_1 \geq 1/4$ ,  $s_2 \geq 0$  and  $b > 1/2$  we have*

$$\|u\|_{L_x^4 L_\delta^\infty} \lesssim \|u\|_{X_{s_1, b}^\rho}, \quad (3.11)$$

$$\|u\|_{L_x^8 L_\delta^8} \lesssim \|u\|_{X_{s_2, b}^\rho}, \quad (3.12)$$

$$\|u\|_{L_x^6 L_\delta^6} \lesssim \|u\|_{X_{s_2, b}^\rho}. \quad (3.13)$$

*Proof.* The inequalities (3.11) and (3.12), follows from

$$\|U(t)u_0\|_{L_x^4 L_\delta^\infty} \lesssim \|u_0\|_{H^{1/4}}, \quad \|U(t)u_0\|_{L_x^8 L_\delta^8} \lesssim \|u_0\|_{L^2}$$

and from a standard argument, see for example [2, 10, 14].

The inequality (3.13) follows by interpolation between  $\|v\|_{L_x^8 L_\delta^8} \lesssim \|v\|_{X_{0, 1/2+}}$  and the trivial estimate  $\|v\|_{L_x^2 L_\delta^2} \lesssim \|v\|_{0,0}$ .  $\square$

**Remark 3.6.** *Actually the inequality (3.8) is valid for all  $s > -1/4$  (see [5]).*

#### 4. ESTIMATES FOR $\delta_4$ AND $\delta_6$

From here onwards we will consider the notation  $|\xi_i| = N_i$ ,  $m(N_i) = m_i$ ,  $|\xi_{ij}| = N_{ij}$ ,  $m(N_{ij}) = m_{ij}$ , etc. Given four numbers  $N_1, N_2, N_3, N_4$  and  $\mathcal{C} = \{N_1, \dots, N_4\}$ , we will note  $N_s = \max \mathcal{C}$ ,  $N_a = \max \mathcal{C} \setminus \{N_s\}$ ,  $N_t = \max \mathcal{C} \setminus \{N_s, N_a\}$ ,  $N_b = \min \mathcal{C}$ , in this way

$$N_s \geq N_a \geq N_t \geq N_b.$$

**Proposition 4.1.** *Let  $m$  defined as in (3.1), then*

$$|\delta_4| \lesssim m^2(N_s) \quad (4.1)$$

and

$$|\delta_6| \lesssim N_s m^2(N_s). \quad (4.2)$$

In order to prove (4.1) we will use the following proposition in similar form like when  $m = 1$ .

**Proposition 4.2.** *Let  $m$  defined as in (3.1), then*

$$|\xi_2 \xi_4^2 m_4^2 + \xi_4 \xi_2^2 m_2^2 + \xi_1 \xi_3^2 m_3^2 + \xi_3 \xi_1^2 m_1^2| \lesssim m^2(N_s) |\xi_{12} \xi_{13} \xi_{14}|, \quad (4.3)$$

and

$$|\xi_1 \xi_4^2 m_4^2 + \xi_4 \xi_1^2 m_1^2 + \xi_2 \xi_3^2 m_3^2 + \xi_3 \xi_2^2 m_2^2| \lesssim m^2(N_s) |\xi_{12} \xi_{13} \xi_{14}|. \quad (4.4)$$

*Proof.* Without loss of generality we can assume  $|\xi_1| = N_s$ , and by symmetry  $|\xi_{12}| \leq |\xi_{14}|$ . In [10] (Lemma 4.1) they proved that

$$|\xi_2 \xi_4^2 m_4^2 + \xi_4 \xi_2^2 m_2^2 + \xi_1 \xi_3^2 m_3^2 + \xi_3 \xi_1^2 m_1^2| \lesssim m^2(N_s) |\xi_{12} \xi_{14}| N_s,$$

therefore we can suppose  $|\xi_{13}| \ll N_s$ , this implies  $|\xi_1| \sim |\xi_3|$ . Let  $f(\xi) = \xi m(\xi)$ , observing that  $\xi_{12}\xi_{14} = \xi_2\xi_4 - \xi_1\xi_3$ , we have

$$\begin{aligned} \xi_2\xi_4^2m_4^2 + \xi_4\xi_2^2m_2^2 + \xi_1\xi_3^2m_3^2 + \xi_3\xi_1^2m_1^2 &= \xi_2\xi_4(f(\xi_2) + f(\xi_4)) + \xi_1\xi_3(f(\xi_1) + f(\xi_3)) \\ &= \xi_2\xi_4(f(\xi_1) + f(\xi_2) + f(\xi_3) + f(\xi_4)) - \xi_{12}\xi_{14}(f(\xi_1) + f(\xi_3)). \end{aligned} \quad (4.5)$$

In the second term of (4.5) we can use the medium value theorem (MVT) for to obtain

$$|\xi_{12}\xi_{14}(f(\xi_1) + f(\xi_3))| = |\xi_{12}\xi_{14}(f(\xi_1) - f(-\xi_3))| \lesssim |\xi_{12}\xi_{14}\xi_{13}|m^2(N_s),$$

where we used that  $|\xi_{13}| \ll N_s$ , and  $|f'(\xi_1)| \sim |m^2(\xi_1)|$ . Therefore we will only estimate the first term in (4.5).

We consider two cases:

1)  $|\xi_{14}| \gtrsim |\xi_3|$ , in this case we consider two sub-cases

a) If  $|\xi_{12}| \ll |\xi_1|$ , then using the DMVT (Lemma 3.1) with  $\xi = -\xi_1$ ,  $\lambda = \xi_{12}$  and  $\eta = \xi_{13}$

$$|\xi_2\xi_4(f(\xi_1) + f(\xi_2) + f(\xi_3) + f(\xi_4))| \lesssim |\xi_{14}|N_s|\xi_{12}\xi_{13}|\frac{m^2(N_s)}{N_s},$$

where we also used that  $|\xi_2| \leq |\xi_1| \sim |\xi_3| \lesssim |\xi_{14}|$  and  $|f''(\xi_1)| \lesssim m^2(\xi_1)/|\xi_1|$ .

b) If  $|\xi_{12}| \gtrsim |\xi_1|$ , here we proceed similarly as in [10] (Lemma 4.1). Using the fact that  $N_s \lesssim |\xi_{12}| \leq |\xi_{14}|$ ,  $(m^2(\xi)\xi^2)' \sim m^2(\xi)\xi$ ,  $m^2(\xi)\xi$  is nondecreasing and the MVT we have

$$\begin{aligned} |\xi_2\xi_4^2m_4^2 + \xi_4\xi_2^2m_2^2 + \xi_1\xi_3^2m_3^2 + \xi_3\xi_1^2m_1^2| &= |\xi_3(m_1^2\xi_1^2 - m_{1-13}^2\xi_{1-13}^2) + \xi_2(m_4^2\xi_4^2 \\ &\quad - m_{4+13}^2\xi_{4+13}^2) - \xi_{24}(m_3^2\xi_3^2 - m_{4+13}^2\xi_{4+13}^2)| \lesssim |\xi_{13}|N_s^2m^2(N_s). \end{aligned}$$

Hence we obtain (4.3) in this sub-case.

2)  $|\xi_{14}| \ll |\xi_3|$ , using the TMVT considering in Lemma 3.1:  $\xi = -\xi_1$ ,  $\lambda = \xi_{12}$ ,  $\eta = \xi_{13}$  and  $\gamma = \xi_{14}$  we have

$$|\xi_2\xi_4(f(\xi_1) + f(\xi_2) + f(\xi_3) + f(\xi_4))| \lesssim N_s^2|\xi_{12}\xi_{13}\xi_{14}|\frac{m^2(N_s)}{N_s^2},$$

where we also used that  $|f'''(\xi_1)| \lesssim m^2(\xi_1)/|\xi_1|^2$ .

Now, in order to obtain (4.4), using (4.3) we get

$$|\xi_1\xi_4^2m_4^2 + \xi_4\xi_1^2m_1^2 + \xi_2\xi_3^2m_3^2 + \xi_3\xi_2^2m_2^2| \lesssim m^2(N_s)|\xi_{21}\xi_{23}\xi_{24}| = m^2(N_s)|\xi_{12}\xi_{14}\xi_{13}|.$$

This completes the proof.  $\square$

By (2.5), (2.8) and Proposition 4.2, we have (4.1). The estimate (4.2) is obvious.

## 5. ESTIMATES 4-LINEAL AND 6-LINEAL

The following lemma will be used frequently in the estimates 4-lineal and 6-lineal.

**Lemma 5.1.** *Let  $n \geq 2$  a even integer,  $w_1, \dots, w_n \in \mathbf{S}(\mathbb{R})$ , then*

$$\int_{\xi_1 + \dots + \xi_n = 0} \widehat{w}_1 \widehat{w}_2 \dots \widehat{w}_{n-1} \widehat{w}_n = \int_{\mathbb{R}} w_1 \overline{w}_2 \dots w_{n-1} \overline{w}_n. \quad (5.1)$$

In the proof of our global result, we will need the following properties.

**Proposition 5.2.** *Let  $w \in \mathbf{S}(\mathbb{R} \times \mathbb{R})$ , then we have*

$$\left| \int_0^\rho \Lambda_6(\delta_6; w(t)) dt \right| \lesssim N^{-3} \|Iw\|_{X_{1,1/2+}^\rho}^6 \quad (5.2)$$

and

$$|\Lambda_4(\delta_4; w(t))| \lesssim \|Iw\|_{H^1}^4. \quad (5.3)$$

*Proof.* As in [9, 10, 11], we first perform a Littlewood-Paley decomposition of the six factors  $w$ , so that the  $\xi_i$  are essentially the constants  $N_i$ ,  $i = 1, \dots, 6$ . To recover the sum at the end we borrow a  $N_s^{-\epsilon}$  from the large denominator  $N_s$  and often this will not be mentioned. Also without loss of generality we can assume that the Fourier transforms in the left-side of (5.2) and (5.3) are real and nonnegative.

Let  $I = \{s, a, t, b\}$  the set of indices such that  $N_s \geq N_a \geq N_t \geq N_b$ . We will proved first (5.2), we divide the proof into two cases.

1)  $N_b \gtrsim N$ , by definition of  $m$  we have  $N_t m_t \gtrsim N$  and  $N_b m_b \gtrsim N$ , therefore

$$N_s m_s^2 \lesssim N^{-3} N_s m_s N_a m_a N_t m_t N_b m_b,$$

and consequently by (5.1), Hölder inequality, (3.3) and Lemma 3.5, we have

$$\begin{aligned} \left| \int_0^\rho \Lambda_6(\delta_6; w(t)) dt \right| &\lesssim N^{-3} \int_0^\rho \int_{\mathbb{R}} \prod_{j \in I} D_x I w_j \prod_{j \notin I} w_j dx dt \\ &\lesssim N^{-3} \prod_{j \in I} \|D_x I w_j\|_{L_x^6 L_t^\rho} \prod_{j \notin I} \|w_j\|_{L_x^6 L_t^\rho} \\ &\lesssim N^{-3} \prod_{j \in I} \|I w_j\|_{X_{1,1/2+}^\rho} \prod_{j \notin I} \|I w_j\|_{X_{1-s,1/2+}^\rho} \\ &\lesssim N^{-3} \|Iw\|_{X_{1,1/2+}^\rho}^6. \end{aligned}$$

2)  $N_b \ll N$ , by (2.6) and Proposition 2.2, if  $N_s \ll N$ , then  $\Lambda_6(\delta_6) = 0$ , therefore we can assume  $N_s \gtrsim N$ , and for  $\xi_1 + \dots + \xi_6 = 0$  this implies  $N_s \sim N_a \gtrsim N$ , hence

$$N_s m_s^2 \lesssim N^{-1} N_s m_s N_a m_a,$$

by (5.1), Hölder inequality, (3.3) and Lemmas 3.5 and 3.2 one obtains

$$\begin{aligned}
 \left| \int_0^\rho \Lambda_6(\delta_6; w(t)) dt \right| &\lesssim N^{-1} \int_0^\rho \int_{\mathbb{R}} D_x I w_s D_x I w_a \prod_{j \notin \{s, a\}} w_j dx dt \\
 &\lesssim N^{-1} \|(D_x I w_s) w_b\|_{L_x^4 L_\rho^2} \|(D_x I w_a) w_p\|_{L_x^4 L_\rho^2} \|w_t\|_{L_x^4 L_\rho^\infty} \|w_q\|_{L_x^4 L_\rho^\infty} \\
 &\lesssim N^{-3} \|I w_s\|_{X_{1,1/2+}^\rho} \|w_b\|_{X_{s_0,1/2+}^\rho} \|I w_a\|_{X_{1,1/2+}^\rho} \|w_p\|_{X_{s_0,1/2+}^\rho} \\
 &\quad \|w_t\|_{X_{1/4,1/2+}^\rho} \|w_q\|_{X_{1/4,1/2+}^\rho} \\
 &\lesssim N^{-3} \|I w\|_{X_{1,1/2+}^\rho}^6,
 \end{aligned}$$

where  $1/4 < s_0 < 1$ .

For to prove (5.3), by (4.1) and (5.1) we have

$$\begin{aligned}
 |\Lambda_4(\delta_4; w(t))| &\lesssim \int_{\xi_1 + \dots + \xi_n} \delta_4(\xi_1, \dots, \xi_n) \widehat{w}_1 \widehat{w}_2 \widehat{w}_3 \widehat{w}_4 \\
 &\lesssim \int_{\mathbb{R}} |w(t)|^4 dx \lesssim \|w(t)\|_{H^{1/4}}^4 \\
 &\lesssim \|I w\|_{H^1}^4.
 \end{aligned}$$

Which finished the proof.  $\square$

## 6. PROOF OF THEOREM 1.2

We will use the following results.

**Lemma 6.1.** *If  $u$  is a solution of IVP (1.1), then*

$$\|I u(t)\|_{L^2} \leq \|I \varphi\|_{H^{1-s}}.$$

for  $0 \leq s < 1$ .

*Proof.* The lemma follows from definition of  $I$ , the conservation law in  $L^2$  and definition of  $l(\xi)$ .  $\square$

**Lemma 6.2.** *If  $u$  is a solution of IVP (1.1), then*

$$|E_2(t) - E_1(t)| \leq c \|I \varphi\|_{H^1}^4 + c E_1(t)^4. \quad (6.1)$$

If  $k$  is a positive integer and  $u(t)$  is defined in the time interval  $[0, k]$ , then

$$E_2(k) = E_1(0) + \Lambda_4(\delta_4)(0) + \sum_{j=1}^k \int_{j-1}^j \Lambda_6(\delta_6)(t) dt. \quad (6.2)$$

*Proof.* The inequality (6.1) is obvious from (2.3), (5.3) and Lemma 6.1.

By (2.6) we have

$$E_2(k) = E_2(0) + \sum_{j=1}^k \int_{j-1}^j \Lambda_6(\delta_6)(t) dt.$$

and by (2.3) we obtain (6.2).  $\square$

**6.1. Rescaling.** We know that if  $u(x, t)$  is a solution of (1.5) with initial data  $u(x, 0) = \varphi$ , then

$$u_\lambda(x, t) = \frac{1}{\lambda} u\left(\frac{x}{\lambda}, \frac{t}{\lambda^3}\right),$$

is also a solution of (1.5) with initial data

$$u_\lambda(x, 0) = \frac{1}{\lambda} u\left(\frac{x}{\lambda}, 0\right) = \frac{1}{\lambda} \varphi\left(\frac{x}{\lambda}\right) := \varphi_\lambda.$$

Let  $c_0 \in (0, 1)$  a constant to be chosen later, we have

$$\begin{aligned} \|I\varphi_\lambda\|_{H^1} &\sim \|\partial_x I\varphi_\lambda\|_{L^2} + \|I\varphi_\lambda\|_{L^2} \\ &\lesssim \frac{N^{1-s}}{\lambda^{1/2+s}} \|D_x^s \varphi\|_{L^2} + \frac{1}{\sqrt{\lambda}} \|\varphi\|_{L^2} \\ &< c_0. \end{aligned}$$

taking

$$\lambda \sim N^{\frac{2(1-s)}{1+2s}} \left( \frac{\|D_x^s \varphi\|_{L^2}}{c_0} \right)^{\frac{2}{1+2s}} \quad \text{and} \quad N > \left( \frac{\|\varphi\|_{L^2}}{c_0} \right)^{\frac{2s}{1-s}}. \quad (6.3)$$

**6.2. Iteration.** Without lost of generality we can assume  $k_1 = 1$  in (2.2). We consider our solution rescaled with initial data

$$\|I\varphi\|_{H^1} = \epsilon_0 < c_0 < 1,$$

then by Theorem 3.3 we have a solution of (1.1) defined in the time interval  $[0, 1]$ . For to extend the solution of local theorem in the time interval  $[0, \lambda^3 T]$  we need to prove that  $\|Iu(n)\|_{H^1} \lesssim \epsilon_0$ , for all  $n \in \{1, 2, \dots, m_{\lambda, T}\} = W$ , where  $m_{\lambda, T} \sim \lambda^3 T$ . Indeed we will prove that

$$\|Iu(n)\|_{H^1}^2 \leq 3\epsilon_0^2, \quad n \in W, \quad (6.4)$$

but as  $\|Iu(t)\|_{H^1}^2 = \|Iu(t)\|_{L^2} + \|\partial_x Iu(t)\|_{L^2}$ , by Lemma 6.1 is sufficient to prove

$$\|\partial_x Iu(n)\|_{L^2}^2 \leq 2\epsilon_0^2, \quad n \in W. \quad (6.5)$$

We will prove (6.5) by induction.

1) When  $k = 1$ , we suppose by contradiction that  $\|\partial_x Iu(1)\|_{L^2}^2 > 2\epsilon_0^2$ , then there exist  $t_0 \in (0, 1)$  such that  $\|\partial_x Iu(t_0)\|_{L^2}^2 = 2\epsilon_0^2$ , from (6.1) we have

$$|E_2(t_0) - 2\epsilon_0^2| \leq 5c\epsilon_0^4,$$

and using (2.3), (2.6), (3.6) and (5.2) we obtain

$$E_2(t_0) = E_1(0) + \Lambda_4(\delta_4)(0) + \int_0^{t_0} \Lambda_6(\delta_6)(0),$$

and from here

$$|E_2(t_0) - \epsilon_0^2| \leq 5c\epsilon_0^4 + \frac{1}{N^3} 8c\epsilon_0^6,$$



hence if  $\epsilon_0^2 < \frac{1}{20c}$ , we have

$$\epsilon_0^2 \leq |2\epsilon_0^2 - E_2(t_0)| + |E_2(t_0) - \epsilon_0^2| \leq 5c\epsilon_0^4 + 8c\epsilon_0^6 + 5c\epsilon_0^4 < \epsilon_0^2,$$

but this is a contradiction.

2) Now, we suppose (6.5) for  $n = 1, 2, \dots, k$ , with  $k \geq 2$  a positive integer, then we also will prove (6.5) for  $n = k + 1$ . In fact, in similar way as in case 1), we suppose by contradiction that  $\|\partial_x Iu(k+1)\|_{L^2}^2 > 2\epsilon_0^2$ , then there exist  $t_0 \in (0, k+1)$  such that  $\|\partial_x Iu(t_0)\|_{L^2}^2 = 2\epsilon_0^2$ . Similarly as in the case 1), from (6.1) we have

$$|E_2(t_0) - 2\epsilon_0^2| \leq 5c\epsilon_0^4, \tag{6.6}$$

by (2.6) and (6.2) we get

$$|E_2(t_0) - E_1(0)| \leq |\Lambda_4(\delta_4)(0)| + \left| \sum_{j=1}^{[t_0]} \int_{j-1}^j \Lambda_6(\delta_6)(t) dt \right| + \left| \int_{[t_0]}^{t_0} \Lambda_6(\delta_6)(t) dt \right|,$$

therefore by (3.6) and (5.2) we easily deduce that

$$\begin{aligned} |E_2(t_0) - \epsilon_0^2| &\leq 5c\epsilon_0^4 + (1 + [t_0]) \frac{8c}{N^3} \epsilon_0^6 \\ &\leq 5c\epsilon_0^4 + \lambda^3 T \frac{8c}{N^3} \epsilon_0^6. \end{aligned} \tag{6.7}$$

As in the case  $k = 1$ , by (6.6) and (6.7) we obtain a contradiction if  $\lambda^3 T \sim N^3$ , consequently we can to iterate this process  $m_{\lambda, T} \sim \lambda^3 T$  times if  $T \sim \lambda^{-3} N^3$  and by (6.3) if

$$T \sim N(12s - 3)/(1 + 2s).$$

Hence  $u$  is globally well-posed in  $H^s$  for all  $s > 1/4$ .

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