

A PRIORI ESTIMATES FOR SOLUTIONS OF A NONLINEAR DISPERSIVE EQUATION

XAVIER CARVAJAL

ABSTRACT. In this work we obtain some a priori estimates for a higher order Schrödinger equation and in particular we obtain some a priori estimates for the modified Korteweg-de Vries equation.

1. INTRODUCTION

In this paper we will describe some results on the growth of Sobolev norms for solutions of the initial value problem (IVP)

$$\begin{cases} \partial_t u + ia \partial_x^2 u + b \partial_x^3 u + ic |u|^2 u + d |u|^2 \partial_x u + e u^2 \partial_x \bar{u} = 0, & x, t \in \mathbb{R}, \\ u(x, 0) = u_0(x), \end{cases} \quad (1.1)$$

where u is a complex valued function and a, b, c, d and e are real parameters with $be \neq 0$.

This model was proposed by Hasegawa and Kodama in [11, 14] to describe the nonlinear propagation of pulses in optical fibers. In literature, this model is called as a higher order nonlinear Schrödinger equation or also Airy-Schrödinger equation.

We consider the following gauge transformation

$$v(x, t) = \exp\left(i\lambda x + i(a\lambda^2 - 2b\lambda^3)t\right) u\left(x + (2a\lambda - 3b\lambda^2)t, t\right), \quad (1.2)$$

then, u solves (1.1) if and only if v satisfies the IVP

$$\begin{cases} \partial_t v + i(a - 3\lambda b)\partial_x^2 v + b\partial_x^3 v + i(c - \lambda(d - e))|v|^2 v + d|v|^2 \partial_x v + e v^2 \partial_x \bar{v} = 0, \\ v(x, 0) = \exp(i\lambda x) u(x, 0). \end{cases} \quad (1.3)$$

Thus, if we take $\lambda = a/3b$ in (1.2) and $c = (d - e)a/3b$, then the function

$$v(x, t) = \exp\left(i\frac{a}{3b}x + i\frac{a^3}{27b^2}t\right) u\left(x + \frac{a^2}{3b}t, t\right), \quad (1.4)$$

satisfies the complex modified Korteweg-de Vries type equation (complex mKdV)

$$\begin{cases} \partial_t v + b\partial_x^3 v + d|v|^2 \partial_x v + e v^2 \partial_x \bar{v} = 0, \\ v(x, 0) = \exp(iax/3b) u(x, 0). \end{cases} \quad (1.5)$$

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It was shown in [15] that the flow associated to the IVP (1.1) leaves the following quantity

$$I_1(u) = \int_{\mathbb{R}} |u|^2(x, t) dx, \quad (1.6)$$

conserved in time. Also, when $be \neq 0$ we have the following conserved quantity

$$I_2(u) = k_1 \int_{\mathbb{R}} |\partial_x u|^2(x, t) dx + k_2 \int_{\mathbb{R}} |u|^4(x, t) dx + k_3 \operatorname{Im} \int_{\mathbb{R}} u(x, t) \partial_x \overline{u(x, t)} dx, \quad (1.7)$$

where $k_1 = 3be$, $k_2 = -e(e+d)/2$ and $k_3 = (3bc - a(e+d))$. We may suppose $k_3 = 0$. In fact, when $k_3 \neq 0$ we can take in the gauge transformation (1.2) $\lambda = -\frac{k_3}{6bc}$. Then, u solves (1.1) if and only if v satisfies (1.3) and in this new IVP we have the constant $k_3 = 0$. The conserved quantity (1.7) with $k_3 = 0$, gives

$$\int_{\mathbb{R}} |\partial_x u|^2(x, t) dx + c \int_{\mathbb{R}} |u|^4(x, t) dx = \int_{\mathbb{R}} |\partial_x u|^2(x, 0) dx + c \int_{\mathbb{R}} |u|^4(x, 0) dx. \quad (1.8)$$

The main result in this work is

Theorem 1.1. *Let $u \in \mathcal{C}(\mathbb{R}, H^1)$ be the solution of IVP (1.5) and $0 \leq \theta < 1/2$, then*

$$\|D_x^\theta u(t)\|_{L_x^2} \leq c \|D_x^\theta u(0)\|_{L_x^2} \exp \{ct(\|D_x^{1/4} u(0)\|_{L_x^2}^4 + \|u(0)\|_{L_x^2}^6 + 1)\}, \quad (1.9)$$

where c is a constant.

2. GROWTH OF SOBOLEV NORMS

In the following proposition we obtain a sharp estimate for $|\|u_x(t)\|_{L^2} - \|u_x(0)\|_{L^2}|$, we will need the following elemental lemma

Lemma 2.1. *Let $x \geq 0$, $a \geq 0$ and $b \geq 0$, if $x^2 - ax - b \leq 0$ then*

$$0 \leq x \leq a + b^{1/2}. \quad (2.10)$$

Proposition 2.2. *Let $u_0 \in H^1$ and $u(t)$ the solution of IVP (1.1) with $k_3 = 0$, then*

$$|\|u_x(t)\|_{L^2} - \|u_x(0)\|_{L^2}| \leq c \|u(0)\|_{L^2}^3.$$

Proof. If $c < 0$ in (1.8), Gagliardo-Nirenberg's inequality yields

$$\begin{aligned} \int_{\mathbb{R}} |\partial_x u|^2(x, t) dx &\leq |c| \int_{\mathbb{R}} |u|^4(x, t) dx + \int_{\mathbb{R}} |\partial_x u|^2(x, 0) dx \\ &\leq |c| \|u(0)\|_{L^2}^3 \|u_x(t)\|_{L^2} + \int_{\mathbb{R}} |\partial_x u|^2(x, 0), \end{aligned}$$

therefore by (2.10)

$$\|u_x(t)\|_{L^2} \leq \|u_x(0)\|_{L^2} + |c| \|u(0)\|_{L^2}^3.$$

If $c > 0$ in (1.8), applying Gagliardo-Nirenberg's inequality we have

$$\begin{aligned} \int_{\mathbb{R}} |\partial_x u|^2(x, t) dx &\leq c \int_{\mathbb{R}} |u|^4(x, 0) dx + \int_{\mathbb{R}} |\partial_x u|^2(x, 0) dx \\ &\leq c \|u(0)\|_{L^2}^3 \|u_x(0)\|_{L^2} + \int_{\mathbb{R}} |\partial_x u|^2(x, 0), \end{aligned} \quad (2.11)$$

similarly we get

$$\begin{aligned} \int_{\mathbb{R}} |\partial_x u|^2(x, 0) dx &\leq c \int_{\mathbb{R}} |u|^4(x, t) dx + \int_{\mathbb{R}} |\partial_x u|^2(x, t) dx \\ &\leq c \|u(0)\|_{L^2}^3 \|u_x(t)\|_{L^2} + \int_{\mathbb{R}} |\partial_x u|^2(x, t), \end{aligned}$$

this implies that

$$\begin{aligned} \|u_x(0)\|_{L^2} &\leq c^{1/2} \|u(0)\|_{L^2}^{3/2} \|u_x(t)\|_{L^2}^{1/2} + \|u_x(t)\|_{L^2} \\ &\leq \frac{c}{2} \|u(0)\|_{L^2}^3 + \frac{1}{2} \|u_x(t)\|_{L^2} + \|u_x(t)\|_{L^2}, \end{aligned} \quad (2.12)$$

using (2.12) in (2.11) it follows that

$$\|u_x(t)\|_{L^2}^2 \leq c \|u(0)\|_{L^2}^3 \left(\frac{c}{2} \|u(0)\|_{L^2}^3 + \frac{3}{2} \|u_x(t)\|_{L^2} \right) + \|u_x(0)\|_{L^2}^2, \quad (2.13)$$

and again using (2.10) we obtain

$$\|u_x(t)\|_{L^2} \leq \|u_x(0)\|_{L^2} + c \|u(0)\|_{L^2}^3.$$

Analogously when $c \in \mathbb{R} \setminus \{0\}$, by symmetry we get

$$\|u_x(0)\|_{L^2} \leq \|u_x(t)\|_{L^2} + c \|u(0)\|_{L^2}^3.$$

This concludes the proof of the proposition. \square

Lemma 2.3. *Let $u(x, t) = U(t)u_0(x)$ be the solution of the IVP (1.1) with $c = d = e = 0$, we have the following local smoothing effect*

$$\|\partial_x U(t)u_0\|_{L_x^\infty L_T^2} \leq c \|u_0\|_{L^2}, \quad (2.14)$$

and the dual version

$$\left\| \partial_x \int_0^t U(t-t') f(t', x) dt' \right\|_{L_x^2} \leq c \|f\|_{L_x^1 L_t^2}. \quad (2.15)$$

We have also the maximal function estimate

$$\|U(t)u_0\|_{L_x^4 L_T^\infty} \leq c \|u_0\|_{H^{1/4}}. \quad (2.16)$$

And

$$\left\| \partial_x^2 \int_0^t U(t-t') f(t', x) dt' \right\|_{L_x^\infty L_t^2} \leq c \|f\|_{L_x^1 L_t^2}. \quad (2.17)$$

Proof. For the proof of the inequalities (2.14)-(2.16) see [4] or [12] or [20]. For the proof of inequality (2.17) see [12]. \square

The following lemmas were proved in [12].

Lemma 2.4. *Let be $\theta \in (0, 1)$, $1 < p_1, p_2, q_1, q_2 < \infty$, such that $1/p_1 + 1/p_2 = 1$, $1/q_1 + 1/q_2 = 1/2$, then*

$$\|D_x^\theta(uv) - uD_x^\theta v - vD_x^\theta u\|_{L_x^1 L_t^2} \leq c \|u\|_{L_x^{p_1} L_t^{q_1}} \|D_x^\theta v\|_{L_x^{p_2} L_t^{q_2}}. \quad (2.18)$$

Lemma 2.5. *Let be $\theta \in (0, 1)$, $1 < p_1, p_2, p, r, r_2 < \infty$, $r_1 \in (1, \infty]$ such that $1/p_1 + 1/p_2 = 1/p$, $1/r_1 + 1/r_2 = 1/r$ then*

$$\|D_x^\theta F(f)\|_{L_x^p L_t^r} \leq c \|F'(f)\|_{L_x^{p_1} L_t^{r_1}} \|D_x^\theta f\|_{L_x^{p_2} L_t^{r_2}}. \quad (2.19)$$

Without loss of generality we will restringed our attention, for the modified Korteweg-de Vries equation, mKdV ($\alpha = c = d = 0$ and $u_0(x) \in \mathbb{R}$ in (1.1)).

Initially we considered the generalized KdV equation

$$\begin{cases} \partial_t u + \partial_x^3 u + \partial_x u^k = 0, & x, t \in \mathbb{R}, k \geq 2, k \in \mathbb{Z}^+, \\ u(x, 0) = u_0(x), \end{cases} \quad (2.20)$$

and the equivalent integral equality

$$u(t) = U(t)u_0 - \int_0^t U(t-t') \partial_x u^k(t') dt'. \quad (2.21)$$

The IVP (2.20) have the following conserved quantities

$$I_1(u) = \|u(t)\|_{L_x^2}^2, \quad I_2(u) = \|u_x(t)\|_{L_x^2}^2 + c_k \|u(t)\|_{L_x^{k+1}}^{k+1}. \quad (2.22)$$

Theorem 2.6. *Let $u \in \mathcal{C}(\mathbb{R}, H^1)$ be the solution of IVP (2.20) and $0 \leq \theta < 1/2$, then*

$$\|D_x^\theta u(t)\|_{L_x^2}^2 \leq c \|D_x^\theta u(0)\|_{L_x^2}^2 + c \|u(t')\|_{L_x^{2(k-1)} L_t^\infty}^{2(k-1)} \int_0^t \|D_x^\theta u(t')\|_{L_x^2}^2 dt'. \quad (2.23)$$

Proof. A straightforward calculation yields

$$\int_0^t \int_{\mathbb{R}} D_x^\theta u [D_x^\theta u_t + D_x^\theta (u^k)_x] = \int_0^t \int_{\mathbb{R}} D_x^\theta u D_x^\theta [u_t + u_{xxx} + (u^k)_x] = 0,$$

therefore

$$2 \int_0^t \int_{\mathbb{R}} D_x^\theta u D_x^\theta u_t = \int_{\mathbb{R}} D_x^\theta u(t)^2 - \int_{\mathbb{R}} D_x^\theta u(0)^2 = -2 \int_0^t \int_{\mathbb{R}} D_x^\theta u D_x^\theta (u^k)_x := 2J$$

and using (2.21) and Lemma 2.3 we obtain

$$\begin{aligned} J &= \int_0^t \int_{\mathbb{R}} \partial_x U(t) D_x^\theta u_0 D_x^\theta u^k + \int_0^t \int_{\mathbb{R}} \partial_x^2 \int_0^{t'} U(t'-\tau) D_x^\theta u^k(\tau) d\tau D_x^\theta u^k \\ &\leq c \|\partial_x U(t) D_x^\theta u_0\|_{L_x^\infty L_t^2} \|D_x^\theta u^k\|_{L_x^1 L_t^2} + \left\| \partial_x^2 \int_0^{t'} U(t'-\tau) D_x^\theta u^k(\tau) d\tau \right\|_{L_x^\infty L_t^2} \|D_x^\theta u^k\|_{L_x^1 L_t^2} \\ &\leq c \|D_x^\theta u_0\|_{L_x^2} \|D_x^\theta u^k\|_{L_x^1 L_t^2} + c \|D_x^\theta u^k\|_{L_x^1 L_t^2} \|D_x^\theta u^k\|_{L_x^1 L_t^2} \\ &\leq c \|D_x^\theta u_0\|_{L_x^2}^2 + c \|D_x^\theta u^k\|_{L_x^1 L_t^2}^2 \\ &\leq c \|D_x^\theta u_0\|_{L_x^2}^2 + c \|u(t')\|_{L_x^{2(k-1)} L_t^\infty}^{2(k-1)} \int_0^t \|D_x^\theta u\|_{L_x^2}^2, \end{aligned}$$

where in the last inequality we used (2.18) and (2.19) (see Remark 2.7), thus

$$\|D_x^\theta u(t)\|_{L_x^2}^2 \leq c\|D_x^\theta u(0)\|_{L_x^2}^2 + c\|u(t')\|_{L_x^{2(k-1)}L_t^\infty}^{2(k-1)} \int_0^t \|D_x^\theta u(t')\|_{L_x^2}^2 dt'. \quad \square$$

If $k = 3$ (mKdV) in (2.23) we have

$$\|D_x^\theta u(t)\|_{L_x^2}^2 \leq c\|D_x^\theta u(0)\|_{L_x^2}^2 + c\|u(t')\|_{L_x^4L_t^\infty}^4 \int_0^t \|D_x^\theta u(t')\|_{L_x^2}^2 dt'. \quad (2.24)$$

But if $u \in \mathcal{C}(\mathbb{R}, H^1)$, then $u(t')$ is continuous in $[0, t]$, therefore $\|u(t')\|_{L_x^4L_t^\infty} = \|u(t_0)\|_{L_x^4}$ for some $t_0 \in [0, t]$, then we have

$$\|u(t')\|_{L_x^4L_t^\infty} = \|u(t_0)\|_{L_x^4} \leq \|u(t')\|_{L_t^\infty L_x^4} \leq \|u(t')\|_{L_x^4L_t^\infty}. \quad (2.25)$$

Hence by (2.24) and (2.25) we obtain

$$\|D_x^\theta u(t)\|_{L_x^2}^2 \leq c\|D_x^\theta u(0)\|_{L_x^2}^2 + c\|u(t')\|_{L_t^\infty L_x^4}^4 \int_0^t \|D_x^\theta u(t')\|_{L_x^2}^2 dt'. \quad (2.26)$$

Remark 2.7. 1) The inequality (2.18) is valid with $2 < q_1, q_2 < \infty$, $1/q_1 + 1/q_2 = 1/2$, therefore for all $q_1 > 2$, $k \geq 2$ and $0 \leq \theta < 1/2$ we get

$$\begin{aligned} \|D_x^\theta u^k\|_{L_x^1L_t^2} &\leq c\|u^{k-1}\|_{L_x^2L_t^{q_1}} \|D_x^\theta u\|_{L_x^2L_t^{\frac{2}{(1-2/q_1)}}} + \|u\|_{L_x^{2(k-1)}L_t^\infty} \|D_x^\theta u^{k-1}\|_{L_x^{\frac{2(k-1)}{(2k-3)}L_t^2}L_t^2} \\ &\leq ct^{1/q_1} \|u^{k-1}\|_{L_x^2L_t^\infty} \|D_x^\theta u\|_{L_x^2L_t^{\frac{2}{(1-2/q_1)}}} + c\|u\|_{L_x^{2(k-1)}L_t^\infty} \|u^{k-2}\|_{L_x^{\frac{2(k-1)}{(k-2)}L_t^2}L_t^2} \|D_x^\theta u\|_{L_x^2L_t^2} \\ &\leq ct^{1/q_1} \|u\|_{L_x^{2(k-1)}L_t^\infty}^{k-1} \|D_x^\theta u\|_{L_x^2L_t^{\frac{2}{(1-2/q_1)}}} + c\|u\|_{L_x^{2(k-1)}L_t^\infty}^{k-1} \|D_x^\theta u\|_{L_x^2L_t^2}. \end{aligned} \quad (2.27)$$

Now if $\theta \in [0, 1/2)$, $u \in \mathcal{C}(\mathbb{R}, H^1)$ and $\tau_n \rightarrow \tau$, then

$$|D_x^\theta u(\tau_n, \xi) - D_x^\theta u(\tau, \xi)| \leq c\|u(\tau_n) - u(\tau)\|_{H^1} \rightarrow 0,$$

thus $D_x^\theta u(\tau, \xi)$ is continuous in the variable τ , and in similar way as in (2.25) we have $\|D_x^\theta u(t')\|_{L_x^2L_t^\infty} = \|D_x^\theta u(t')\|_{L_t^\infty L_x^2}$. Hence taking the limit when $q_1 \rightarrow \infty$ in (2.27), Lebesgue's dominated convergence theorem yields

$$\|D_x^\theta u^k\|_{L_x^1L_t^2} \leq c\|u\|_{L_x^{2(k-1)}L_t^\infty}^{k-1} \|D_x^\theta u\|_{L_x^2L_t^2}.$$

2) Seeking (2.22) and (2.23) we observe that $2(k-1) = k+1$ if and only if $k = 3$ (mKdV).

Theorem 2.8. Let $u \in \mathcal{C}(\mathbb{R}, H^1)$ be the solution of IVP (2.20) with $k = 3$ and $0 \leq \theta < 1/2$, then

$$\|D_x^\theta u(t)\|_{L_x^2} \leq c\|D_x^\theta u(0)\|_{L_x^2} \exp \left\{ ct(\|D_x^{1/4}u(0)\|_{L_x^2}^4 + \|u(0)\|_{L_x^2}^6 + 1) \right\}. \quad (2.28)$$

Proof. We consider two cases:

Case 1) : if $\|D_x u(0)\|_{L_x^2} \leq 1$.

From (1.8), Proposition 2.2 and the Sobolev embedding $\|v\|_{L_x^4} \leq c\|v\|_{H^{1/4}}$, it follows that

$$\begin{aligned} \int u^4(x, t) &\leq \int u^4(x, 0) + c\|D_x u(t)\|_{L_x^2}^2 + c\|D_x u(0)\|_{L_x^2}^2 \\ &\leq \int u^4(x, 0) + c\left(2\|D_x u(0)\|_{L_x^2}^2 + c\|u(0)\|_{L_x^2}^6\right) + c\|D_x u(0)\|_{L_x^2}^2 \\ &\leq c\left(\|D_x^{1/4} u(0)\|_{L_x^2}^4 + \|u(0)\|_{L_x^2}^6 + 1\right). \end{aligned} \quad (2.29)$$

The inequality (2.28) is a direct consequence of (2.26), (2.29) and Gronwall's inequality.

Case 2) : if $\|D_x u(0)\|_{L_x^2} > 1$.

By scaling we consider the solution

$$u_\lambda(x, t) = \lambda u(\lambda x, \lambda^3 t), \quad \lambda = \frac{1}{\|D_x u(0)\|_{L_x^2}^{2/3}}.$$

We have

$$\|D_x^s u_\lambda(t)\|_{L_x^2} = \lambda^{1/2+s} \|D_x^s u(\lambda^3 t)\|_{L_x^2}, \quad (2.30)$$

therefore

$$\|D_x u_\lambda(0)\|_{L_x^2} = \lambda^{3/2} \|D_x u(0)\|_{L_x^2} = 1,$$

hence by Case 1) we get

$$\|D_x^\theta u_\lambda(t)\|_{L_x^2} \leq c\|D_x^\theta u_\lambda(0)\|_{L_x^2} \exp\left\{ct\left(\|D_x^{1/4} u_\lambda(0)\|_{L_x^2}^4 + \|u_\lambda(0)\|_{L_x^2}^6 + 1\right)\right\}. \quad (2.31)$$

And using (2.30) we obtain for all $t \in \mathbb{R}$:

$$\begin{aligned} \|D_x^\theta u(\lambda^3 t)\|_{L_x^2} &\leq c\|D_x^\theta u(0)\|_{L_x^2} \exp\left\{ct\left(\lambda^3 \|D_x^{1/4} u(0)\|_{L_x^2}^4 + \lambda^3 \|u(0)\|_{L_x^2}^6 + 1\right)\right\} \\ &\leq c\|D_x^\theta u(0)\|_{L_x^2} \exp\left\{ct\left(\|D_x^{1/4} u(0)\|_{L_x^2}^4 + \|u(0)\|_{L_x^2}^6 + 1\right)\right\}. \end{aligned}$$

This completes the proof of the lemma. \square

In the next section we prove a refined a priori estimate.

3. GROWTH OF SOBOLEV NORMS OF LOW FREQUENCY SOLUTIONS

In this section we will prove the following theorem

Theorem 3.1. *Let $u_0 \in H^s$, $s \geq 1/4$, $v_0(x) = (\chi_{\{|\xi| < N\}} \widehat{u}_0)^\vee(x)$ and $v(t)$ a solution of IVP (1.1) with initial data v_0 and $k_3 = 0$, then there exist a $\theta_0 = \theta_0(u_0, v(t)) \in (0, 1)$ such that for all $\theta \in [0, 1]$*

$$\|v(t)\|_{H^\theta} \leq \|v(0)\|_{H^\theta} + \frac{\langle N \rangle^{\theta_0}}{\sqrt{e\theta_0 \log \langle N^2 \rangle}} P(\|v(0)\|_{L^2}), \quad (3.32)$$

where $P(x)$ is a polinomio of grau three.

To prove Theorem 3.1 we need the following result

Lemma 3.2. *Let $v \in H^1$, then for $\theta \in [0, 1)$ we have*

$$\|v\|_{H^\theta}^2 \leq \|v\|_{L^2}^2 \exp \left\{ \frac{\theta \int (1 + \xi^2)^\theta \log(1 + \xi^2) |\widehat{v}(\xi)|^2 d\xi}{\|v\|_{H^\theta}^2} \right\}. \quad (3.33)$$

Proof. Let $F(\theta) = \|v\|_{H^\theta}^2 \neq 0$ for $\theta \in [0, 1)$ and $G(\theta) = F'(\theta)/F(\theta)$, then

$$F'(\theta) = \int (1 + \xi^2)^\theta \log(1 + \xi^2) |\widehat{v}(\xi)|^2 d\xi, \quad F''(\theta) = \int (1 + \xi^2)^\theta \log^2(1 + \xi^2) |\widehat{v}(\xi)|^2 d\xi$$

and using Cauchy-Schwartz

$$F'(\theta) \leq F(\theta)^{1/2} F''(\theta)^{1/2}.$$

Thus

$$G'(\theta) = \frac{F(\theta)F''(\theta) - F'(\theta)^2}{F(\theta)^2} \geq 0.$$

Hence $G(\theta)$ is increasing. Now we define $H(\theta) = \log F(\theta)$, using the mean value theorem for $\theta \in (0, 1)$ we get

$$\log F(\theta) - \log F(0) \leq \theta G(\gamma\theta) \leq \theta G(\theta) = \theta \frac{F'(\theta)}{F(\theta)},$$

where $\gamma \in (0, 1]$. Therefore

$$F(\theta) \leq F(0) \exp \left\{ \frac{\theta F'(\theta)}{F(\theta)} \right\}.$$

□

Proof of Theorem 3.1. If $\theta = 0$ or $\theta = 1$ is a obvious consequence of (1.6) and Proposition 2.2. Now for t fixed, we define the function

$$f_t(\delta) := \|v(t)\|_{H^\delta}^2 - \|v(0)\|_{H^\delta}^2, \quad \delta \in [0, 1],$$

and we consider

$$f_t(\delta_0) = \max_{\delta \in [0, 1]} f_t(\delta).$$

If $\delta_0 = 0$ or $\delta_0 = 1$, we have (3.32). If $\delta_0 \in (0, 1)$ then $f_t'(\delta_0) = 0$, therefore

$$\int (1 + \xi^2)^{\delta_0} \log(1 + \xi^2) |\widehat{v}(t, \xi)|^2 d\xi = \int (1 + \xi^2)^{\delta_0} \log(1 + \xi^2) |\widehat{v}(0, \xi)|^2 d\xi. \quad (3.34)$$

From (3.33) and (3.34) we have

$$\begin{aligned}
\|v(t)\|_{H^{\delta_0}}^2 &\leq \|v(0)\|_{L^2}^2 \exp \left\{ \frac{\delta_0 \int (1 + \xi^2)^{\delta_0} \log(1 + \xi^2) |\widehat{v}(t, \xi)|^2 d\xi}{\|v(t)\|_{H^{\delta_0}}^2} \right\} \\
&= \|v(0)\|_{L^2}^2 \exp \left\{ \frac{\delta_0 \int (1 + \xi^2)^{\delta_0} \log(1 + \xi^2) |\widehat{v}(0, \xi)|^2 d\xi}{\|v(t)\|_{H^{\delta_0}}^2} \right\} \\
&\leq \|v(0)\|_{L^2}^2 \exp \left\{ \frac{\delta_0 \log(1 + N^2) \int (1 + \xi^2)^{\delta_0} |\widehat{v}(0, \xi)|^2 d\xi}{\|v(t)\|_{H^{\delta_0}}^2} \right\} \\
&= \|v(0)\|_{L^2}^2 (1 + N^2)^{\frac{\delta_0 \|v(0)\|_{H^{\delta_0}}^2}{\|v(t)\|_{H^{\delta_0}}^2}}.
\end{aligned}$$

Let $\chi(t) = \|v(t)\|_{H^{\delta_0}}^2$, $L_0 = \|v(0)\|_{L^2}^2$ and $q = \langle N^2 \rangle^{\delta_0}$, then

$$\frac{\chi(t)}{L_0} \leq q^{\frac{\chi(0)}{\chi(t)}},$$

taking logarithmo in the above inequality we have

$$\begin{aligned}
\frac{\chi(t)}{L_0} \log \frac{\chi(t)}{L_0} &\leq \frac{\chi(0)}{L_0} \log q \\
&= \frac{\frac{\chi(0)}{L_0} \log q}{\left(\frac{\chi(0)}{L_0} + k_0\right) \log \left\{\frac{\chi(0)}{L_0} + k_0\right\}} \left(\frac{\chi(0)}{L_0} + k_0\right) \log \left\{\frac{\chi(0)}{L_0} + k_0\right\}, \quad (3.35)
\end{aligned}$$

where we consider $k_0 = q/(e \log q)$, let us now the function

$$f(x) = (x + k_0) \log\{x + k_0\} - x \log q, \quad x \geq 0,$$

this function have a minimum $x_{min} = q/e - k_0 > 0$ in $[0, \infty)$, equal to $f(x_{min}) = -q/e + k_0 \log q = 0$, and by inequality (3.35) this implies

$$\frac{\chi(t)}{L_0} \log \frac{\chi(t)}{L_0} \leq \left(\frac{\chi(0)}{L_0} + k_0\right) \log \left\{\frac{\chi(0)}{L_0} + k_0\right\}.$$

As the function $g(x) = x \log x$, $x > 0$ is no decreasing, by the definition of δ_0 we obtain (3.32). We concludes the proof of the theorem. \square

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IMECC-UNICAMP, CAMPINAS, 13081-970, BRAZIL
 E-mail address: carvajal@ime.unicamp.br