Frequency decay for Navier-Stokes stationary solutions

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Abstract

We consider stationary Navier-Stokes equations in $\mathbb{R}^3$ with a regular external force and we prove exponential frequency decay of the solutions. Moreover, if the external force is small enough, we give a pointwise exponential frequency decay for such solutions. If a damping term is added to the equation, a pointwise decay is obtained without the smallness condition over the force.

Résumé

Décroissance fréquentielle pour les équations de Navier-Stokes stationnaires. Pour une force extérieure quelconque, mais suffisamment régulière, on démontre la décroissance fréquentielle des solutions de ces équations. Si de plus la force est petite, on peut décrire ponctuellement cette décroissance. La condition de petitesse de la force peut être supprimée si l'on rajoute un terme d'amortissement.

Version française abrégée

Dans cette note on s'intéresse aux équations de Navier-Stokes stationnaires (1) avec une force extérieure indépendante du temps. Si cette dernière est suffisamment régulière (Gevrey régulière), on montre dans le Théorème 1.2 que les solutions de ce problème stationnaire possèdent une décroissance exponentielle en variable de Fourier. Dans le cas d'une force extérieure *petite* (qui correspond à un cadre laminaire), il est possible d'être plus précis et dans le Théorème 1.3 nous donnons avec l'inégalité (4) une estimation ponctuelle pour décrire la décroissance du spectre de ces solutions stationnaires. Si de plus on considère un terme d'amortissement (voir équation (5)), on récupère avec le Théorème 1.4 l'estimation ponctuelle sans condition de petitesse sur la force extérieure. Ces résultats étendent au cadre de l'espace tout entier et à des forces plus générales quelques travaux précédents concernant la régularité Gevrey des solutions des équations de Navier-Stokes stationnaires, voir [1], [6], [2] et [4].

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1. Introduction

Gevrey regularity for solutions of the Navier-Stokes equations has been studied in many different frameworks: for example, in the periodic setting with external force (see [1], [6]) for the stationary problem in $\mathbb{T}^3$ with frequentially localized forces see [2]. For the evolution problem in $\mathbb{R}^3$ (with a null force) a pointwise analysis is obtained in [4]. In this note we generalize some of these previous results in the framework of stationary Navier-Stokes equations in $\mathbb{R}^3$

$$-\nu\Delta\overrightarrow{U} + \mathbb{P}(div(\overrightarrow{U}\otimes\overrightarrow{U})) = \overrightarrow{F}, \qquad div(\overrightarrow{U}) = 0, \quad div(\overrightarrow{F}) = 0, \tag{1}$$

where $\nu > 0$ is the fluid's viscosity parameter, $\overrightarrow{U} : \mathbb{R}^3 \longrightarrow \mathbb{R}^3$ is the velocity, $\mathbb{P}$ is Leray's projector and $\overrightarrow{F} : \mathbb{R}^3 \longrightarrow \mathbb{R}^3$ is a time-independent external force.

Existence of a solution for large data $\overrightarrow{F} \in \dot{H}^{-1}(\mathbb{R}^3)$ goes back to Leray, Finn and Ladyzhenskaya (see [3]) and is based on a priori energy estimates :

Lemma 1.1 *If $\overrightarrow{F} \in \dot{H}^{-1}(\mathbb{R}^3)$ with $div(\overrightarrow{F}) = 0$, then there exists at least one solution $\overrightarrow{U} \in \dot{H}^1(\mathbb{R}^3)$ to the stationary Navier-Stokes equation (1) such that $\nu\|\overrightarrow{U}\|_{\dot{H}^1} \leq \|\overrightarrow{F}\|_{\dot{H}^{-1}}$.*

For a proof see [5], Theorem 16.2.

In this note we study the behavior of such solutions if the external force is regular enough we prove in Theorem 1.2 an exponential frequency decay. Moreover, if the external force is small enough, we give in Theorem 1.3 a pointwise exponential frequency decay for such solutions. Finally, if a damping term is added to the equation, a pointwise decay is obtained in Theorem 1.4 without the smallness condition over the force.

Theorem 1.2 *Let $\overrightarrow{F} \in \dot{H}^{-1}(\mathbb{R}^3)$ with $div(\overrightarrow{F}) = 0$ be such that for some $\varepsilon_0 > 0$ we have*

$$\int_{\mathbb{R}^3} e^{2\varepsilon_0|\xi|}\left|\widehat{\overrightarrow{F}}(\xi)\right|^2|\xi|^{-2}d\xi < +\infty.$$

Then for every solution $\overrightarrow{U}$ to the stationary Navier-Stokes equations (1), such that $\overrightarrow{U} \in \dot{H}^1(\mathbb{R}^3)$, the following exponential frequency decay holds for some $\varepsilon_1 > 0$ (depending on $\overrightarrow{U}$) :

$$\int_{\mathbb{R}^3} e^{2\varepsilon_1|\xi|}\left|\widehat{\overrightarrow{U}}(\xi)\right|^2|\xi|^2d\xi < +\infty. \tag{2}$$

In the laminar setting we obtain a sharper *pointwise* exponential frequency decay.

Theorem 1.3 *There exists a (small) constant $\eta > 0$ such that if $\overrightarrow{F} \in \dot{H}^{-2}\cap\dot{H}^{-1}(\mathbb{R}^3)$ with $\|\overrightarrow{F}\|_{\dot{H}^{-1}}\|\overrightarrow{F}\|_{\dot{H}^{-2}} < \eta$, $div(\overrightarrow{F}) = 0$ and if*

$$\sup_{\xi\in\mathbb{R}^3}\frac{1}{|\xi|}e^{\varepsilon_0|\xi|}\left|\widehat{\overrightarrow{F}}(\xi)\right| \leq c_0 < +\infty, \tag{3}$$

for some $c_0, \varepsilon_0 > 0$, then there exists a solution $\overrightarrow{U} \in L^2\cap\dot{H}^1(\mathbb{R}^3)$ to the stationary Navier-Stokes equations (1) such that $\overrightarrow{U}$ verifies the following pointwise exponential frequency decay for constants $c_1, \varepsilon_1 > 0$ that depend on $\overrightarrow{U}$:

$$\left|\widehat{\overrightarrow{U}}(\xi)\right| \leq c_1e^{-\varepsilon_1|\xi|}|\xi|^{-1}, \quad \textit{for all} \quad \xi \neq 0. \tag{4}$$

If a damping term is added to the stationary Navier-Stokes system, we have the following result.

Theorem 1.4 *Let $\overrightarrow{F} \in H^{-1}(\mathbb{R}^3)$ with $div(\overrightarrow{F}) = 0$ and for $\alpha > 0$ let us consider the damped stationary Navier-Stokes equations*

$$-\nu\Delta\overrightarrow{U} + \mathbb{P}(div(\overrightarrow{U} \otimes \overrightarrow{U})) = \overrightarrow{F} - \alpha\overrightarrow{U}, \qquad div(\overrightarrow{U}) = 0. \tag{5}$$

If the external force $\overrightarrow{F}$ satisfies the frequency decay (3), for some $c_0, \varepsilon_0 > 0$, then every stationary solution $\overrightarrow{U} \in H^1(\mathbb{R}^3)$ satisfies the following pointwise exponential frequency decay for some $c_1, \varepsilon_1 > 0$ that depend on $\overrightarrow{U}$:

$$\left|\widehat{\overrightarrow{U}}(\xi)\right| \leq c_2|\xi|e^{-\varepsilon_1|\xi|}. \tag{6}$$

2. Proof of Theorem 1.2

Lemma 2.1 *Let $T_0 > 0$. For $\overrightarrow{u}_0 \in \dot{H}^1(\mathbb{R}^3)$ a divergence-free initial data and $\overrightarrow{f} \in \mathcal{C}([0, T_0[, \dot{H}^1(\mathbb{R}^3))$ a divergence-free external force there exists a time $0 < T_1 < T_0$ and a function $\overrightarrow{u} \in \mathcal{C}([0, T_1[, \dot{H}^1(\mathbb{R}^3))$ which is a unique solution to the Navier-Stokes equations*

$$\partial_t\overrightarrow{u} - \nu\Delta\overrightarrow{u} + \mathbb{P}(div(\overrightarrow{u} \otimes \overrightarrow{u})) = \overrightarrow{f}, \quad div(\overrightarrow{u}) = 0, \qquad \overrightarrow{u}(0, \cdot) = \overrightarrow{u}_0. \tag{7}$$

Existence and uniqueness issues are classical, see [5] for details. In the following proposition we prove the frequency decay for the solution $\overrightarrow{u}$ obtained in Lemma 2.1.

Proposition 2.1 *Let $\beta > 0$ and consider the Poisson kernel $e^{\beta\sqrt{t}\sqrt{-\Delta}}$. Within the framework of Lemma 2.1, if the external force $\overrightarrow{f}$ is such that $e^{\beta\sqrt{t}\sqrt{-\Delta}}\overrightarrow{f} \in \mathcal{C}(]0, T_0[, \dot{H}^1(\mathbb{R}^3))$ then the unique solution of equations (7) satisfies $e^{\beta\sqrt{t}\sqrt{-\Delta}}\overrightarrow{u} \in \mathcal{C}(]0, T_1[, \dot{H}^1(\mathbb{R}^3))$ for all time $t \in [0, T_1[$ where $0 < T_1 < T_0$ is small enough.*

Proof. Consider the space $E = \left\{\overrightarrow{u} \in \mathcal{C}(]0, T_1[, \dot{H}^1(\mathbb{R}^3)) : e^{\beta\sqrt{t}\sqrt{-\Delta}}\overrightarrow{u} \in \mathcal{C}(]0, T_1[, \dot{H}^1(\mathbb{R}^3))\right\}$, endowed with the norm $\|\cdot\|_E = \|e^{\beta\sqrt{t}\sqrt{-\Delta}}(\cdot)\|_{L^\infty_t \dot{H}^1_x}$. We study the quantity

$$\|\overrightarrow{u}_1\|_E = \left\|h_{\nu t} * \overrightarrow{u}_0 + \int_0^t h_{\nu(t-s)} * \overrightarrow{f}(s, \cdot)ds - \int_0^t h_{\nu(t-s)} * \mathbb{P}(div(\overrightarrow{u}_1 \otimes \overrightarrow{u}_1))(s, \cdot)ds\right\|_E, \tag{8}$$

where $h_{\nu t}$ is the heat kernel. The two first terms of this expression are easy to estimate and we have

$$\left\|h_{\nu t} * \overrightarrow{u}_0 + \int_0^t h_{\nu(t-s)} * \overrightarrow{f}(s, \cdot)ds\right\|_E \leq c(\nu, \beta, T_0)\left(\|\overrightarrow{u}_0\|_{\dot{H}^1_x} + \|e^{\beta\sqrt{t}\sqrt{-\Delta}}\overrightarrow{f}\|_{L^\infty_t \dot{H}^1_x}\right). \tag{9}$$

For the last term of (8), by definition of the norm $\|\cdot\|_E$, by the Plancherel formula and by the boundedness of the Leray projector we have

$$\begin{aligned}(I) = \left\|\int_0^t h_{\nu(t-s)} * \mathbb{P}(div(\overrightarrow{u}_1 \otimes \overrightarrow{u}_1))ds\right\|_E &= \sup_{0<t<T_1}\left\|e^{\beta\sqrt{t}\sqrt{-\Delta}}\left(\int_0^t h_{\nu(t-s)} * \mathbb{P}(div(\overrightarrow{u}_1 \otimes \overrightarrow{u}_1))ds\right)\right\|_{\dot{H}^1_x} \\ &\leq \sup_{0<t<T_1} c\left\||\xi|^2\int_0^t e^{-\nu(t-s)|\xi|^2}e^{\beta\sqrt{t}|\xi|}\left|(\mathcal{F}[\overrightarrow{u}_1] * \mathcal{F}[\overrightarrow{u}_1])(s, \cdot)\right|ds\right\|_{L^2_x}.\end{aligned}$$

Since we have the pointwise inequality

$$e^{\beta\sqrt{t}|\xi|}\left|(\mathcal{F}[\overrightarrow{u}_1]*\mathcal{F}[\overrightarrow{u}_1])(s,\xi)\right| \leq \left[\left(e^{\beta\sqrt{t}|\xi|}|\mathcal{F}[\overrightarrow{u}_1]|\right)*\left(e^{\beta\sqrt{t}|\xi|}|\mathcal{F}[\overrightarrow{u}_1]|\right)\right](s,\xi), \tag{10}$$

due to the fact that $e^{\beta\sqrt{t}|\xi|} \leq e^{\beta\sqrt{t}|\xi-\eta|}e^{\beta\sqrt{t}|\eta|}$ for all ξ, $\eta\in\mathbb{R}^3$, then we obtain

$$(I) \leq \sup_{0<t<T_1} c\int_0^t \left\| |\xi|^{\frac{3}{2}} e^{-\nu(t-s)|\xi|^2}|\xi|^{\frac{1}{2}}\left|\left[\left(e^{\beta\sqrt{t}|\xi|}|\mathcal{F}[\overrightarrow{u}_1]|\right)*\left(e^{\beta\sqrt{t}|\xi|}|\mathcal{F}[\overrightarrow{u}_1]|\right)\right]\right|\right\|_{L^2_x} ds.$$

Getting back to the spatial variable we can write

$$\begin{aligned}
(I) &\leq \sup_{0<t<T_1} c\int_0^t \left\|(-\Delta)^{\frac{3}{4}}h_{\nu(t-s)} * (-\Delta)^{\frac{1}{4}}\left\{\left(\mathcal{F}^{-1}\left[e^{\beta\sqrt{t}|\xi|}|\mathcal{F}[\overrightarrow{u}_1]|\right]\right)\otimes\left(\mathcal{F}^{-1}\left[e^{\beta\sqrt{t}|\xi|}|\mathcal{F}[\overrightarrow{u}_1]|\right]\right)\right\}\right\|_{L^2_x} ds \\
&\leq \left(c\int_0^{T_1}\left\|(-\Delta)^{\frac{3}{4}}h_{\nu(t-s)}\right\|_{L^1} ds\right)\left\|\left(\mathcal{F}^{-1}\left[e^{\beta\sqrt{t}|\xi|}|\mathcal{F}[\overrightarrow{u}_1]|\right]\right)\otimes\left(\mathcal{F}^{-1}\left[e^{\beta\sqrt{t}|\xi|}|\mathcal{F}[\overrightarrow{u}_1]|\right]\right)\right\|_{L^\infty_t\dot{H}^{\frac{1}{2}}_x} \\
&\leq c\frac{T^{\frac{1}{4}}}{\nu^{\frac{3}{4}}}\left\|\mathcal{F}^{-1}\left[e^{\beta\sqrt{t}|\xi|}|\mathcal{F}[\overrightarrow{u}_1]|\right]\right\|_{L^\infty_t\dot{H}^1_x}\left\|\mathcal{F}^{-1}\left[e^{\beta\sqrt{t}|\xi|}|\mathcal{F}[\overrightarrow{u}_1]|\right]\right\|_{L^\infty_t\dot{H}^1_x} = c\frac{T_1^{\frac{1}{4}}}{\nu^{\frac{3}{4}}}\|\overrightarrow{u}_1\|_E\|\overrightarrow{u}_1\|_E.
\end{aligned} \tag{11}$$

With estimates (9) and (11) at hand, we fix T_1 small enough in order to apply Picard's contraction principle and we obtain a solution $\overrightarrow{u}_1\in E$ of (7). Since $E\subset\mathcal{C}(]0,T_1[,\dot{H}^1(\mathbb{R}^3))$ we have $\overrightarrow{u}_1\in\mathcal{C}(]0,T_1[,\dot{H}^1(\mathbb{R}^3))$ and by uniqueness of the solution $\overrightarrow{u}$ we have $\overrightarrow{u}_1=\overrightarrow{u}$, and thus $\overrightarrow{u}\in E$. ■

Now, we come back to the stationary Navier-Stokes equations (1) and we will prove that the solution $\overrightarrow{U}\in\dot{H}^1(\mathbb{R}^3)$ (given by Lemma 1.1) satisfies the exponential frequency decay given in (2). In the space $\mathcal{C}(]0,1[,\dot{H}^1(\mathbb{R}^3))$ we consider the evolution problem (7) with the initial data $\overrightarrow{u}_0=\overrightarrow{U}$ where the external force $\overrightarrow{f}$ is now given by with the expression $\overrightarrow{f}=e^{-\beta\sqrt{t}\sqrt{-\Delta}}(e^{\beta\sqrt{t}\sqrt{-\Delta}}\overrightarrow{F})$, for the particular value $\beta=\frac{2}{3}\varepsilon_0>0$ where $\varepsilon_0>0$ is given in the hypothesis of the force $\overrightarrow{F}$. To obtain a unique solution $\overrightarrow{u}\in\mathcal{C}(]0,1[,\dot{H}^1(\mathbb{R}^3))$ to the equations (7) such that $e^{\beta\sqrt{t}\sqrt{-\Delta}}\overrightarrow{u}\in\mathcal{C}(]0,1[,\dot{H}^1(\mathbb{R}^3))$, we prove that the external force $\overrightarrow{f}$ verifies the hypotheses of Lemma 2.1 and Proposition 2.1 above:

$$\begin{aligned}
\left\|e^{\beta\sqrt{t}\sqrt{-\Delta}}\overrightarrow{F}\right\|^2_{L^\infty_t\dot{H}^1_x} &= \sup_{0<t<1}\int_{\mathbb{R}^3}|\xi|^2 e^{2\beta\sqrt{t}|\xi|}\left|\widehat{\overrightarrow{F}}(\xi)\right|^2 d\xi \leq \frac{1}{\beta^4}\int_{\mathbb{R}^3}(\beta|\xi|)^4 e^{2\beta|\xi|}\left|\widehat{\overrightarrow{F}}(\xi)\right|^2|\xi|^{-2}d\xi \\
&\leq \frac{1}{\beta^4}\int_{\mathbb{R}^3}e^{3\beta|\xi|}\left|\widehat{\overrightarrow{F}}(\xi)\right|^2|\xi|^{-2}d\xi = \frac{1}{\beta^4}\int_{\mathbb{R}^3}e^{2\varepsilon_0|\xi|}\left|\widehat{\overrightarrow{F}}(\xi)\right|^2|\xi|^{-2}d\xi < +\infty.
\end{aligned}$$

Thus, once we have $e^{\beta\sqrt{t}\sqrt{-\Delta}}\overrightarrow{F}\in\mathcal{C}(]0,1[,\dot{H}^1(\mathbb{R}^3))$, since the operator $e^{-\beta\sqrt{t}\sqrt{-\Delta}}$ is bounded in the space $\mathcal{C}(]0,1[,\dot{H}^1(\mathbb{R}^3))$ we have $\overrightarrow{f}=e^{-\beta\sqrt{t}\sqrt{-\Delta}}(e^{\beta\sqrt{t}\sqrt{-\Delta}}\overrightarrow{F})\in\mathcal{C}(]0,1[,\dot{H}^1(\mathbb{R}^3))$. Moreover, we have $e^{\beta\sqrt{t}\sqrt{-\Delta}}\overrightarrow{f}=e^{\beta\sqrt{t}\sqrt{-\Delta}}\overrightarrow{F}\in\mathcal{C}(]0,1[,\dot{H}^1(\mathbb{R}^3))$. By Lemma 2.1 there exists a time $0<T_1<1$ and a unique solution $\overrightarrow{u}\in\mathcal{C}(]0,T_1[,\dot{H}^1(\mathbb{R}^3))$ to the equation (7). Moreover, since $e^{\beta\sqrt{t}\sqrt{-\Delta}}\overrightarrow{f}\in\mathcal{C}(]0,1[,\dot{H}^1(\mathbb{R}^3))$ by Proposition 2.1 we have $e^{\beta\sqrt{t}\sqrt{-\Delta}}\overrightarrow{u}\in\mathcal{C}(]0,T_1[,\dot{H}^1(\mathbb{R}^3))$. Since the solution $\overrightarrow{U}\in\dot{H}^1(\mathbb{R}^3)$ of the stationary Navier-Stokes equations (1) is a constant in time, we have $\overrightarrow{U}\in\mathcal{C}(]0,T_1[,\dot{H}^1(\mathbb{R}^3))$ and since $\partial_t\overrightarrow{U}\equiv 0$ and $\overrightarrow{f}=e^{-\beta\sqrt{t}\sqrt{-\Delta}}(e^{\beta\sqrt{t}\sqrt{-\Delta}}\overrightarrow{F})=\overrightarrow{F}$ we find that $\overrightarrow{U}\in\mathcal{C}(]0,T_1[,\dot{H}^1(\mathbb{R}^3))$ is also a solution to the equation (7) and thus, by uniqueness we get $\overrightarrow{U}=\overrightarrow{u}$. Then, since $e^{\beta\sqrt{t}\sqrt{-\Delta}}\overrightarrow{u}\in\mathcal{C}(]0,T_1[,\dot{H}^1(\mathbb{R}^3))$ we have $e^{\beta\sqrt{t}\sqrt{-\Delta}}\overrightarrow{U}\in\mathcal{C}(]0,T_1[,\dot{H}^1(\mathbb{R}^3))$ for all time $t\in[0,T_1[$. Thus, if $\varepsilon_1=\beta\sqrt{\frac{T_1}{2}}>0$, we have

$$\int_{\mathbb{R}^3} e^{2\varepsilon_1|\xi|}|\widehat{\vec{U}}(\xi)|^2|\xi|^2 d\xi = \|e^{\beta\sqrt{\frac{T_1}{2}}\sqrt{-\Delta}}\vec{U}\|^2_{\dot{H}^1_x} \leq \sup_{0<t<T_1} \|e^{\beta\sqrt{t}\sqrt{-\Delta}}\vec{U}\|^2_{\dot{H}^1_x} < +\infty,$$

and we obtain the frequency decay given in (2). ■

3. Proof of Theorem 1.3

In the case of a small force, we don't need to use the Scheafer fixed point theorem to get existence of a solution. The condition $\|\vec{F}\|_{\dot{H}^{-1}}\|\vec{F}\|_{\dot{H}^{-2}} < \eta$ ensures that $\|\vec{F}\|_{\dot{H}^{-3/2}}$ is small enough to allow the construction a solution $\vec{U}$ through Picard's iterative scheme, see [5] for the detail. Moreover, this solution will belong to $L^2 \cap \dot{H}^1 = H^1$.

Applying Theorem 1.2, we find that $\vec{U}$ has an exponential decay

$$\int_{\mathbb{R}^3} e^{2\varepsilon_1|\xi|}\left|\widehat{\vec{U}}(\xi)\right|^2|\xi|^2 d\xi \leq c_1 < +\infty.$$

Moreover, we may skip the weight $|\xi|^2$, as $\vec{U}$ belongs to L^2, we have

$$\int_{\mathbb{R}^3} e^{2\varepsilon_1|\xi|}\left|\widehat{\vec{U}}(\xi)\right|^2 d\xi \leq c_3 < +\infty. \tag{12}$$

Now we write

$$\vec{U} = \frac{1}{\nu}\mathbb{P}\left(\frac{1}{\Delta}div(\vec{U}\otimes\vec{U})\right) - \frac{1}{\nu}\frac{1}{\Delta}\vec{F},$$

and take the Fourier transform

$$\left|\widehat{\vec{U}}(\xi)\right| \leq c\frac{1}{\nu|\xi|}|\widehat{\vec{U}} * \widehat{\vec{U}}(\xi)| + \frac{1}{\nu|\xi|^2}|\widehat{\vec{F}}(\xi)|,$$

hence, by (12) and (3) we find

$$\left|\widehat{\vec{U}}(\xi)\right| \leq \frac{1}{\nu|\xi|}e^{-\varepsilon_1|\xi|}\int_{\mathbb{R}^3}\left(e^{\varepsilon_1|\eta|}|\widehat{\vec{U}}(\eta)|\right)\left(e^{\varepsilon_1|\xi-\eta|}|\widehat{\vec{U}}(\xi-\eta)|\right)d\eta + c_0\frac{e^{-\varepsilon_0|\xi|}}{\nu|\xi|} \leq \frac{c_3^2+c_0}{\nu|\xi|}e^{-\varepsilon_2|\xi|},$$

with $\varepsilon_2 = \min(\varepsilon_0, \varepsilon_1) > 0$. ■

4. Proof of Theorem 1.4

For $\alpha > 0$ and under the hypotheses of Theorem 1.4, the existence of solutions of equation (5) in $H^1(\mathbb{R}^3)$ such that

$$\nu\|\vec{\nabla}\otimes\vec{U}\|_2^2 + \alpha\|\vec{U}\|_2^2 \leq \|\vec{F}\|_{\dot{H}^{-1}}\|\vec{\nabla}\otimes\vec{U}\|_2$$

is given by applying the Scheafer fixed point theorem [5].

Following essentially the same lines of Proposition 2.1 and Theorem 1.2 above, we find that $\vec{U}$ has an exponential decay given by

$$\int_{\mathbb{R}^3} e^{2\varepsilon_1|\xi|}\left|\widehat{\vec{U}}(\xi)\right|^2 d\xi \leq c_1 < +\infty. \tag{13}$$

Now from equation (5) we write

$$\overrightarrow{U} = \mathbb{P}\left(\frac{1}{\nu\Delta - \alpha Id}\left[div(\overrightarrow{U}\otimes\overrightarrow{U})\right]\right) - \frac{1}{\nu\Delta - \alpha Id}\left[\overrightarrow{F}\right],$$

and take the Fourier transform

$$|\widehat{\overrightarrow{U}}(\xi)| \leq c\frac{|\xi|}{\nu|\xi|^2+\alpha}|\widehat{\overrightarrow{U}} * \widehat{\overrightarrow{U}}(\xi)| + \frac{1}{\nu|\xi|^2+\alpha}|\widehat{\overrightarrow{F}}(\xi)|,$$

hence, by (13) and (3) we find

$$\left|\widehat{\overrightarrow{U}}(\xi)\right| \leq \frac{|\xi|}{\alpha}e^{-\varepsilon_1|\xi|}\int_{\mathbb{R}^3}\left(e^{\varepsilon_1|\eta|}|\widehat{\overrightarrow{U}}(\eta)|\right)\left(e^{\varepsilon_1|\xi-\eta|}|\widehat{\overrightarrow{U}}(\xi-\eta)|\right)d\eta + c_0\frac{|\xi|\,e^{-\varepsilon_0|\xi|}}{\alpha} \leq \frac{c_1^2+c_0}{\alpha}|\xi|e^{-\varepsilon_2|\xi|},$$

with $\varepsilon_2 = \min(\varepsilon_0, \varepsilon_1) > 0$. ■

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