Sobolev-like cones of trace-class operators on unbounded domains: interpolation inequalities and compactness properties

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Abstract

In this paper we extend the compactness properties for trace-class operators obtained by Dolbeault, Felmer & Mayorga-Zambrano to an smooth unbounded domain $\Omega \subseteq \mathbb{R}^d$, $d \geq 3$. We consider V, a non-negative potential on Ω that blows up at infinity, and the normed space $H_V(\Omega) = \left\{u \in H_0^1(\Omega) : \|u\|_V^2 = \int_\Omega (|\nabla u(x)|^2 + |u(x)|^2 V(x))\, dx < \infty\right\}$. A positive self-adjoint trace-class operator R belongs to the Sobolev-like cone $\mathscr{H}^1_{V,+}$ if $(\psi_{i,R})_{\mathbb{N}} \subseteq H_V(\Omega)$ and $\langle\langle R \rangle\rangle_V = \sum_{i=1}^{\infty} \nu_{i,R} \|\psi_{i,R}\|_V^2 < \infty$, where $(\nu_{i,R})_{i\in\mathbb{N}}$ is the sequence of occupation numbers of R and $(\psi_{i,R})_{i\in\mathbb{N}} \subseteq L^2(\Omega)$ is a corresponding Hilbertian base of eigenfunctions. We prove that a sequence in $\mathscr{H}^1_{V,+}$, bounded in energy $\langle\langle \cdot \rangle\rangle_V$, has a subsequence that converges in trace norm; this is analogous to the classical Sobolev immersion $\mathrm{H}^1(\Omega) \subseteq \mathrm{L}^2(\Omega)$. We prove the existence of lower bounds for non-linear free energy functionals and, by doing so, we establish Lieb-Thirring type inequalities as well some Gagliardo-Nirenberg type interpolation inequalities; then our compactness result is applied to minimize non-linear free energy functionals working on $\mathscr{H}^1_{V,+}$.

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1. Introduction

Self-adjoint positive trace-class operators $R : L^2(\Omega) \to L^2(\Omega)$, $\Omega \subseteq \mathbb{R}^d$, appear quite naturally in the Heisenberg picture of Quantum Mechanics (see e.g. [1]). By the Riesz-Schauder and Hilbert-Schmidt theorems, there exist a sequence of eigenvalues $(\nu_{i,R})_{i\in\mathbb{N}} \subseteq \mathbb{R}^+_*$ and a Hilbertian base of eigenfunctions $(\psi_{i,R})_{i\in\mathbb{N}} \subseteq L^2(\Omega)$. Because of their interpretation in Physics, an eigenvalue $\nu_{i,R}$ is usually refered as an *occupation*

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number and the corresponding eigenfunction $\psi_{i,R}$ is refered as a *wave function*; a *mixed state* is a pair $(\nu_{i,R}, \psi_{i,R})_{i\in\mathbb{N}}$ (see e.g. [4] and [10]).

Along this work we shall assume that $\Omega \subseteq \mathbb{R}^d$ is an unbounded domain, $d \geq 3$, and that the operators R are such that the corresponding eigenfunctions belong to the Sobolev space $H_0^1(\Omega)$ and it's energy

$$\sum_{i=1}^{\infty} |\nu_{i,R}| \left(\int_\Omega |\nabla \psi_{i,R}(x)|^2 + |\psi_{i,R}(x)|^2 V(x)\, dx \right),$$

is finite. Here $V : \Omega \to \mathbb{R}$ is a prescribed non-negative locally-integrable potential verifying

$$\lim_{|x|\to\infty} \operatorname{ess} V(x) = \infty. \tag{1.1}$$

We denote $\mathscr{H}_V^1$ the set of these operators.

In this paper we extend the results of [5] where Ω was assumed bounded. Our main result (Theorem 4.1) is a compactness property for the Solobev-like cone

$$\mathscr{H}_{V,+}^1 = \{L \in \mathscr{H}_V^1 / L \geq 0\},$$

that is, a sequence in $\mathscr{H}_{V,+}^1$, bounded in energy $\langle\langle\cdot\rangle\rangle_V$, has a subsequence that converges in trace-norm to an operator in $\mathscr{H}_{V,+}^1$. As it will be seen, the unboundedness of Ω is compensated by the property (1.1) because it implies the compactness of the immersion $H_V(\Omega) \subseteq L^q(\Omega)$, $q \in [2, 2^*]$ (see Proposition 2.1).

To achieve our goal, we consider a class of non-linear free energy functionals (sometimes called generalized entropy functionals) like

$$\mathscr{F}_{V,\beta}(R) = \operatorname{Tr}((-\Delta + V)R + \beta(R)), \quad R \in \mathscr{H}_{V,+}^1, \tag{1.2}$$

which has been used for a number of applications concerning Partial Differential Equations (see e.g. [10], [15], [7], [8], [3], [9] and [12]). We prove the existence of lower bounds for functionals more general than (1.2) and, by doing so, we establish Lieb-Thirring type inequalities as well some Gagliardo-Nirenberg type interpolation inequalities.

Our setting could physically correspond to an external potential having a singularity, as it's the case of some potentials generated by doping charged impurities in semiconductors. For the relation of the kind of results we obtain with the estimation of the first eigenvalue of the Schrödinger operator $-\Delta + V$ and the analysis of the stability of repulsive Schrödinger-Poisson systems we refer the reader to [4].

This paper is organized as follows. In Section 2 we make a short review of definitions and present the Sobolev-like cone $\mathscr{H}_V^1$ together with some of its properties, in particular, a regularity result (Proposition 2.3) for the density functions associated to operators in $\mathscr{H}_{V,+}^1$. In Section 3, a Casimir class of functions is introduced to define non-linear free energy functionals; then we prove Lieb-Thirring and Gagliardo-Nirenberg type inequalities: Propositions 3.1, 3.2 and 3.3, and Theorem 3.1. Section 4 is dedicated to our main result, Theorem 4.1, that stablishes a compactness property that is analogous to the classical Sobolev immersion but at trace-class operators level. Finally, we use this result to minimize non-linear free energy functionals in Section 5.

2. Definitions and preliminary results

Let $\Omega \subseteq \mathbb{R}^d$ be an unbounded domain, $d \geq 3$, with boundary of class C^1. We denote by $\mathscr{L} = \mathscr{L}(L^2(\Omega))$, the set of bounded linear operators acting on $L^2(\Omega)$. By $\mathscr{I}_\infty = \mathscr{I}_\infty(L^2(\Omega))$ and $\mathscr{S}_\infty = \mathscr{S}_\infty(L^2(\Omega))$ we denote, respectively, the spaces of compact operators and compact self-adjoint operators. We also consider the space of trace-class operators (see e.g. [11])

$$\mathscr{I}_1 = \left\{ R \in \mathscr{L} \, : \, \sum_{i=1}^{\infty} |(\psi_i, R\psi_i)_{L^2(\Omega)}| < \infty \right\} \subseteq \mathscr{I}_\infty, \tag{2.1}$$

where $(\psi_i)_{i\in\mathbb{N}}$ is any Hilbertian base of $L^2(\Omega)$. The trace of an operator $R \in \mathscr{I}_1$ is given by

$$\mathrm{Tr}\,(R) = \sum_{i=1}^{\infty} (\psi_i, R\psi_i)_{L^2(\Omega)}. \tag{2.2}$$

Due to the Riesz-Schauder and Hilbert-Schmidt Theorems (see e.g. [2]), for a given $R \in \mathscr{S}_\infty$, there exists $(\nu_{i,R})_{i\in\mathbb{N}} \subseteq \mathbb{R}$ and a Hilbertian base $(\psi_{i,R})_{i\in\mathbb{N}} \subseteq L^2(\Omega)$ such that

$$R\psi_{i,R} = \nu_{i,R}\psi_{i,R}, \quad \text{for all } i \in \mathbb{N}. \tag{2.3}$$

We shall assume that $(|\nu_{i,R}|)_{i\in\mathbb{N}}$ is ordered, that is

$$|\nu_{i,L}| \geq |\nu_{j,L}|, \quad \text{for all } i,j \in \mathbb{N}, i \leq j;$$

if $\nu_{i,R}$ and $-\nu_{i,R}$ are both eigenvalues, then $-|\nu_{i,R}|$ comes first.

On the space $\mathscr{S}_1 = \mathscr{I}_1 \cap \mathscr{S}_\infty$ the trace norm $\|\cdot\|_1$ is given by

$$\|R\|_1 \equiv \mathrm{Tr}\,(|R|) = \sum_{i=1}^{\infty} |\nu_{i,R}| < \infty. \tag{2.4}$$

We consider a potencial $V : \Omega \to \mathbb{R}$ verifying the conditions

(H1) $V(x) \geq 0$, a.e. $x \in \Omega$,
(H2) $\lim_{|x|\to\infty} \mathrm{ess}\, V(x) = \infty$, and
(H3) $V \in L^1_{\mathrm{loc}}(\Omega)$.

Then we define the linear space

$$H_V(\Omega) = \left\{ u \in H^1_0(\Omega) \, : \, \|u\|_V = \left(\int_\Omega (|\nabla u(x)|^2 + |u(x)|^2 V(x))\, dx \right)^{1/2} < \infty \right\}.$$

It's clear that $\|\cdot\|_V$ is a norm.

The following result about $H_V(\Omega)$ is well known and has been used to study nonlinear Schrödinger equations (see e.g. [6]); here it's presented for completeness (for a proof see e.g [13]).

Proposition 2.1. *Let V be a potential verifying* (H1), (H2) *and* (H3). *The immersion*

$$H_V(\Omega) \subseteq L^q(\Omega), \tag{2.5}$$

is compact for all $q \in [2, 2^*]$, where $\frac{1}{2^*} = \frac{1}{2} - \frac{1}{d}$.

Now we are able to present the cone of operators which we shall work on.

Definition 2.1. Let V be a potential verifying (H1), (H2) and (H3). An operator $R \in \mathscr{S}_1$ is in the Sobolev-like cone $\mathscr{H}_V^1$ if the following conditions hold

$$\psi_{i,R} \in H_V(\Omega), \quad \text{for all } i \in \mathbb{N}; \tag{2.6}$$

$$\langle\langle R \rangle\rangle_V \equiv \sum_{i=1}^{\infty} |\nu_{i,R}| \cdot \|\psi_{i,R}\|_V^2 < \infty. \tag{2.7}$$

We call $\langle\langle R \rangle\rangle_V$ the energy of the operator R and write

$$\mathscr{H}_{V,+}^1 = \left\{ R \in \mathscr{H}_V^1 \,:\, R \geq 0 \right\}. \tag{2.8}$$

Some properties of $\mathscr{H}_V^1$ are summarized in the following result.

Proposition 2.2. *Let V be a potential verifying* (H1), (H2) *and* (H3). *Then*

i) *For any $R \in \mathscr{H}_V^1$ and for any $\alpha \in \mathbb{R}$, we have that $\alpha R \in \mathscr{H}_V^1$ and*

$$\langle\langle \alpha R \rangle\rangle_V = |\alpha| \langle\langle R \rangle\rangle_V.$$

ii) *For any $R \in \mathscr{H}_V^1$ and for any $\alpha \in \mathbb{R}$ we have that $\langle\langle \alpha R \rangle\rangle_V = 0$ if and only if $R = 0$ or $\alpha = 0$.*

iii) *There exists a constant $C > 0$ such that*

$$\|R\|_1 \leq C \langle\langle R \rangle\rangle_V, \quad \textit{for all } R \in \mathscr{H}_V^1.$$

Proof. Points i) and ii) are quite easy. Let's prove iii) for $R \in \mathscr{H}_V^1$. By Proposition 2.1, there exists $C > 0$ such that

$$|\nu_{i,R}| \|\psi_{i,R}\|_{L^2(\Omega)}^2 \leq C |\nu_{i,R}| \|\psi_{i,R}\|_V^2, \quad \text{for all } i \in \mathbb{N},$$

whence, $\|R\|_1 = \sum_{i=1}^{\infty} |\nu_{i,R}| \leq C \langle\langle R \rangle\rangle_V$, since $(\psi_{i,R})_{i \in \mathbb{N}}$ is a Hilbertian base for $L^2(\Omega)$. □

Remark 2.1. Point i) in Proposition 2.2 justifies the term *cone* for $\mathscr{H}_V^1$ and $\mathscr{H}_{V,+}^1$. In other hand point ii) implies that

$$\langle\langle R \rangle\rangle = 0 \quad \text{if and only if} \quad R = 0,$$

which is a property verified by the square of a norm; this helps us to interpret Theorem 4.1 as an analogous of the classical Sobolev immersion but at the level of trace-class operators.

Now let's recall the definition of the density funtion (see e.g. [14]) associated to an operator $R \in \mathscr{I}_\infty$:

$$\rho_R(x) = \sum_{i=1}^{\infty} |\nu_{i,R}||\psi_{i,R}(x)|^2, \quad \text{a.e. } x \in \Omega. \tag{2.9}$$

We have the following regularity result.

Proposition 2.3. *Let V be a potencial verifying* (H1)*,* (H2) *and* (H3)*. For any $R \in \mathscr{H}_V^1$ we have*

$$\rho_R \in W^{1,r}(\Omega) \cap L^q(\Omega), \tag{2.10}$$

for all $q \in [1, d/d-2]$ and all $r \in [1, d/d-1]$.

A proof is provided in [13, Lemma 4.1 and Theorem 4.1] and it is similar to that of [5, Proposition 2.2].

Remark 2.2. We can extend Definition 2.1, Propositions 2.2 and 2.3 as well as a number of the results presented in the following sections to a more general setting. Given $k \in \mathbb{N}$ and $p \in [1, \infty)$, an operator $L \in \mathscr{S}_1$ is in the Sobolev-like cone $\mathscr{W}_V^{k,p}$ if

$$\langle\langle R \rangle\rangle_{V,k,p} \equiv \inf_{\mathscr{B}_R} \sum_{i=1}^{\infty} |\nu_{i,R}| \cdot \|\psi_{i,R}\|_{V,k,p}^p < \infty,$$

where $\mathscr{B}_R$ is the set of eigenbases $(\psi_{i,R})_{i \in \mathbb{N}}$ of $L^2(\Omega)$ verifying

$$\psi_{i,R} \in W_0^{1,p}(\Omega) \cap W_V^{k,p}(\Omega), \quad \text{for all } i \in \mathbb{N},$$

where

$$W_V^{k,p}(\Omega) = \left\{ u \in W^{k,p}(\Omega) : \int_\Omega V(x)\,|u(x)|^p < \infty \right\},$$

$$\|u\|_{V,k,p} = \left(\sum_{j=1}^{k} \|D^j u\|_{L^p(\Omega)}^p + \int_\Omega V(x)\,|u(x)|^p \right)^{1/p}.$$

To finish this section let's mention that whenever $k_1 \leq k_2$ and $1 \leq p \leq q < \infty$, the immersions $\mathscr{W}_V^{k_2,p} \subseteq \mathscr{W}_V^{k_1,p}$, and $\mathscr{W}_V^{k,q} \subseteq \mathscr{W}_V^{k,p}$ are continuous.

3. Free Energy functionals

We start this section by defining the Kinetic and Potential Energy functionals. Then we shall define a class of Casimir functions to introduce Entropy and Free Energy functionals.

Definition 3.1. Let V be a potential verifying (H1), (H2) and (H3). The Kinetic Energy functional is given by

$$\mathscr{K}(R) = \sum_{i=1}^{\infty} \nu_{i,R} \int_\Omega |\nabla \psi_{i,R}(x)|^2\, dx, \qquad R \in \mathscr{H}_{V,+}^1. \tag{3.1}$$

The V-Potential Energy functional is given by

$$\mathscr{P}_V(R) = \mathrm{Tr}\,(VR) = \int_\Omega \rho_R(x)V(x)\,dx, \qquad R \in \mathscr{H}^1_{V,+}. \tag{3.2}$$

Remark 3.1. It's not difficult to see that

$$\langle\langle R\rangle\rangle_V = \mathscr{K}(R) + \mathscr{P}_V(R), \quad \text{for all } R \in \mathscr{H}^1_V. \tag{3.3}$$

Moreover, since formally the Kinetic Energy functional verifies $\mathscr{K}(R) = \mathrm{Tr}\,(-\Delta R)$, we have that the energy is formally given by

$$\langle\langle R\rangle\rangle_V = \mathrm{Tr}\,((-\Delta + V)R).$$

To define Casimir classes we shall need the following kind of conditions. We assume that $\alpha > 0$.

(V_α) The operator $-\alpha\Delta + V$ has a sequence of elements

$$\left\{(\lambda^\alpha_{V,i}, \phi^\alpha_{V,i})\right\}_{i\in\mathbb{N}} \subseteq \mathbb{R} \times H^1_0(\Omega) \cap H^2(\Omega),$$

such that $(\phi^\alpha_{V,i})_{i\in\mathbb{N}}$ is a Hilbertian base of $L^2(\Omega)$ and $(\lambda^\alpha_{V,i})_{i\in\mathbb{N}}$ verifies

$$0 < \lambda^\alpha_{V,1} < \lambda^\alpha_{V,2} \leq \ldots \lambda^\alpha_{V,i} \leq \ldots, \quad i = 2, 3, \ldots.$$

The sequences $(\lambda^\alpha_{V,i})_{i\in\mathbb{N}}$ and $(\phi^\alpha_{V,i})_{i\in\mathbb{N}}$ verify

$$\lim_{i\to\infty} \lambda^\alpha_{V,i} = \infty,$$

and

$$(-\Delta + V)\phi_{V,i} = \lambda_{V,i}\,\phi_{V,i}, \quad \text{for all } i \in \mathbb{N}.$$

In the case of $\alpha = 1$, for each $i \in \mathbb{N}$ we shall write $\lambda_{V,i}$ and $\phi_{V,i}$, instead of $\lambda^1_{V,i}$ and $\phi^1_{V,i}$, respectively.

Definition 3.2. Let $\alpha > 0$ and V a potential verifying (V_α). A function $F : \mathbb{R} \to \mathbb{R} \cup \{+\infty\}$ belongs to the Casimir class $\mathscr{C}^\alpha_V$ if it is convex and

$$\sum_{i=1}^\infty F(\lambda^\alpha_{i,V}) < \infty.$$

In the case of $\alpha = 1$, we shall write $\mathscr{C}_V$.

Example 3.1. Let $\gamma \geq d/2$. The function

$$F(s) = \begin{cases} s^{-\gamma}, & \text{si } s \geq 0; \\ +\infty, & \text{si } s < 0, \end{cases} \tag{3.4}$$

belongs to $\mathscr{C}_0 \cap \mathscr{C}_V$ (see e.g. [5]).

Now we introduce the concept of entropy at operators level.

Definition 3.3. Given $R \in \mathscr{H}^1_{V,+}$ and a convex function $\beta : \mathbb{R} \to \mathbb{R}$ such that $\beta(0) = 0$, we call $\beta-$Entropy of R to the number

$$\mathscr{E}_\beta(R) = \mathrm{Tr}\,[\beta(R)] = \sum_{i=1}^{\infty} \beta(\nu_{i,R}), \tag{3.5}$$

and β is refered as an entropy seed. Moreover if V is a potential such that (H1), (H2), (H3) and (V_α) are satisfied, we define the (V,β)-Free Energy of R by

$$\mathscr{F}_{V,\beta}(R) = \mathscr{E}_\beta(R) + \mathscr{K}(R) + \mathscr{P}_V(R). \tag{3.6}$$

We say that an entropy seed β is generated by the convex function F if

$$\beta(s) = F^*(-s) = \sup_{\lambda\in\mathbb{R}} \{-s\lambda - F(\lambda)\}, \quad s \in \mathbb{R}.$$

Example 3.2. The entropy seed generated by F in (2.3) is

$$\beta(s) = \begin{cases} -(1-m)^{m-1} m^{-m} s^m, & \text{si } s \geq 0; \\ +\infty, & \text{si } s < 0, \end{cases} \tag{3.7}$$

where

$$m = \frac{\gamma}{\gamma+1} \in \left[\frac{d}{d+2}, 1\right).$$

Next, we obtain a lower bound for $\mathscr{F}_{V,\beta}$.

Proposition 3.1. *Let V be a potencial such that* (H1), (H2), (H3) *and* (V_1) *are verified. If β is an entropy seed generated by $F \in \mathscr{C}_V$, then*

$$\mathscr{F}_{\beta,V}(R) \geq -\mathrm{Tr}\,[F(-\Delta + V)], \quad \textit{for all } R \in \mathscr{H}^1_{V,+}. \tag{3.8}$$

Remark 3.2. To prove Proposition 3.1 we proceed like in the proof of [5, Lem. 3.1], but in this case, since Ω is unbounded, it verifies that

$$\int_\Omega |\nabla\phi_{V,i}(x)|^2\,dx = \int_\Omega -\Delta\phi_{V,i}(x)\cdot\phi_{V,i}(x)\,dx, \quad \text{for all } i \in \mathbb{N}, \tag{3.9}$$

due to the condition (V_1). In other hand, when Ω is bounded, (3.9) is a consequence of the Divergence Theorem.

In the same way we obtain the following result. A proof is provided in [13] following the ideas of [5].

Proposition 3.2. *Let $\alpha > 0$ and V a potential verifying* (H1), (H2), (H3) *and* (V_α). *If β is an entropy seed generated by $F \in \mathscr{C}^\alpha_V$, then*

$$\mathscr{E}_\beta(R) + \alpha\mathscr{K}(R) + \mathscr{P}_V(R) \geq -\mathrm{Tr}\,[F(-\alpha\Delta + V)], \quad \textit{for all } R \in \mathscr{H}^1_{V,+}. \tag{3.10}$$

The interpolation result that follows is similar to [4, Th. 15].

Theorem 3.1. *Let V be a potential verifying* (H1), (H2), (H3) *and* (V_1). *Let β be an entropy seed generated by $F \in \mathscr{C}_V$, and $G : \mathbb{R} \to \mathbb{R}$ an strictly convex function such that*

$$\mathrm{Tr}\,[F(-\Delta+V)] \leq \int_\Omega G(V(x))\,dx. \tag{3.11}$$

If τ is a function such that

$$(-G)^*(-s) = -\tau(s), \quad \textit{for all } s \in \mathbb{R}, \tag{3.12}$$

then

$$\mathscr{K}(R) + \mathscr{E}_\beta(R) \geq \int_\Omega \tau(\rho_R(x))\,dx, \quad \textit{for all } R \in \mathscr{H}^1_{V,+}. \tag{3.13}$$

Proof. Let $R \in \mathscr{H}^1_{V,+}$ be arbitrary. Using Proposition 3.1, we have that

$$\mathscr{F}_{\beta,V}(R) = \mathscr{E}_\beta(R) + \mathscr{K}(R) + \mathscr{P}_V(R) \geq -\mathrm{Tr}\,[F(-\Delta+V)],$$

therefore, due to (3.11)

$$\begin{aligned} \mathscr{E}_\beta(R) + \mathscr{K}(R) &\geq -\mathrm{Tr}\,[F(-\Delta+V)] - \mathscr{P}_V(R) \\ &\geq \int_\Omega [-G(V(x)) - \rho_R(x)V(x)]\,dx, \end{aligned}$$

so that (3.13) follows from (3.12). □

To finish this section we introduce a generalized free energy functional and some results about it.

Definition 3.4. Let V be a potential verifying (H1), (H2) and (H3). We say that the operator $-\Delta+V$ is ε-coercive, $\varepsilon \in (0,1]$, if it verifies

$$\lambda_{V,1}^{(1-\varepsilon)} \equiv \sup\{u \in \mathbb{R} : -(1-\varepsilon)\Delta + V \geq u\} > -\infty, \tag{3.14}$$

For $\lambda \leq \lambda_{V,1}^{(1-\varepsilon)}$, the free energy funcional $\mathscr{F}^\lambda_{V,\beta} : \mathscr{H}^1_{V,+} \to \mathbb{R}$ is given by

$$\mathscr{F}^\lambda_{V,\beta}(R) = \mathscr{F}_{V,\beta}(R) - \lambda\|R\|_1.$$

In the sense of operators, relation (3.14) corresponds to $-\Delta+V-\lambda_{V,1}^{(1-\varepsilon)} \geq -\varepsilon\Delta$ whenever Dirichlet boundary conditions are considered.

Proposition 3.3. *Let V be a potential verifying* (H1), (H2), (H3) *and* (V_1). *Suppose that $-\Delta+V$ is $\varepsilon-$coercive, $\varepsilon \in (0,1]$, and β an entropy seed generated by $F \in \mathscr{C}_0^{\varepsilon/2}$. For all $\lambda \leq \lambda_{V,1}^{(1-\varepsilon)}$ and for all $R \in \mathscr{H}^1_{V,+}$ we have*

$$\mathscr{F}^\lambda_{V,\beta}(R) \geq -\mathrm{Tr}\left[F\left(-\frac{\varepsilon}{2}\Delta\right)\right] + \frac{\varepsilon}{2}\mathscr{K}(R). \tag{3.15}$$

Moreover, if $F \in \mathscr{C}^{1-\varepsilon}_{V-\lambda}$, then for all $\lambda \leq \lambda_{V,1}^{(1-\varepsilon)}$ and for all $R \in \mathscr{H}^1_{V,+}$ it verifies

$$\mathscr{F}^\lambda_{V,\beta}(R) \geq -\mathrm{Tr}\,[F(-(1-\varepsilon)\Delta + V - \lambda)]. \tag{3.16}$$

Proof. Let $R \in \mathscr{H}^1_{V,+}$ and $\lambda \leq \lambda^{(1-\varepsilon)}_{V,1}$ be arbitrary. It is clear that

$$\begin{aligned} \mathscr{F}^{\lambda}_{V,\beta}(R) &= \left(\mathscr{E}_\beta(R) + \frac{\varepsilon}{2}\mathscr{K}(R)\right) + \frac{\varepsilon}{2}\mathscr{K}(R) + \\ &+ \left((1-\varepsilon)\mathscr{K}(R) + \mathscr{P}_V(R) - \lambda\|R\|_1\right). \end{aligned} \tag{3.17}$$

Since $F \in \mathscr{C}^{\varepsilon/2}_0$, by Proposition 3.2 we have that

$$\mathscr{E}_\beta(R) + \frac{\varepsilon}{2}\mathscr{K}(R) \geq -\mathrm{Tr}\left[F\left(-\frac{\varepsilon}{2}\Delta\right)\right], \tag{3.18}$$

then, it follows from (3.17) and (3.18) that

$$\begin{aligned} \mathscr{F}^{\lambda}_{V,\beta}(R) &\geq -\mathrm{Tr}\left[F\left(-\frac{\varepsilon}{2}\Delta\right)\right] + \\ &+ \frac{\varepsilon}{2}\mathscr{K}(R) + (1-\varepsilon)\mathscr{K}(R) + \mathscr{P}_V(R) - \lambda\|R\|_1. \end{aligned} \tag{3.19}$$

On other hand, since

$$(1-\varepsilon)\mathscr{K}(R) + \mathscr{P}_V(R) - \lambda\|R\|_1 \geq 0, \tag{3.20}$$

it follows, by (3.19) and (3.20), that

$$\mathscr{F}^{\lambda}_{V,\beta}(R) \geq -\mathrm{Tr}\left[F\left(-\frac{\varepsilon}{2}\Delta\right)\right] + \frac{\varepsilon}{2}\mathscr{K}(R),$$

so that (3.15) is proved.

Now, let's prove (3.16). Once again we use Proposition 3.2, with $\alpha = 1-\varepsilon$ and potential $V-\lambda$ to get

$$\mathscr{E}_\beta(R) + (1-\varepsilon)\mathscr{K}(R) + \mathscr{P}_{V-\lambda}(R) \geq -\mathrm{Tr}[F(-(1-\varepsilon)\Delta + V - \lambda)], \tag{3.21}$$

whence, since $1-\varepsilon \leq 1$, we get

$$\mathscr{E}_\beta(R) + \mathscr{K}(R) + \mathscr{P}_{V-\lambda}(R) \geq \mathscr{E}_\beta(R) + (1-\varepsilon)\mathscr{K}(R) + \mathscr{P}_{V-\lambda}(R). \tag{3.22}$$

Now, by (3.21) and (3.22), we obtain

$$\mathscr{E}_\beta(R) + \mathscr{K}(R) + \mathscr{P}_{V-\lambda}(R) \geq -\mathrm{Tr}[F(-(1-\varepsilon)\Delta + V - \lambda)], \tag{3.23}$$

so that, by (3.23), it follows that

$$\mathscr{F}^{\lambda}_{V,\beta} = \mathscr{E}_\beta(R) + \mathscr{K}(R) + \mathscr{P}_V(R) - \lambda\|R\|_1 \geq -\mathrm{Tr}[F(-(1-\varepsilon)\Delta + V - \lambda)].$$

□

Proposition 3.3 allows us to establish the following important result for a family of operators of $\mathscr{H}^1_{V,+}$ that is bounded in free energy.

Corollary 3.1. *Under the conditions of Proposition 3.3. If $(R_\sigma)_{\sigma\in\Sigma} \subseteq \mathscr{H}^1_{V,+}$ is a family such that $(\mathscr{F}^{\lambda}_{V,\beta}(R_\sigma))_{\sigma\in\Sigma} \subseteq \mathbb{R}$ is bounded, then*

$$(\|R_\sigma\|)_{\sigma\in\Sigma}, \quad (\mathscr{K}(R_\sigma))_{\sigma\in\Sigma}, \quad (\mathscr{E}_\beta(R_\sigma))_{\sigma\in\Sigma} \quad \textit{and} \quad (\mathscr{P}_V(R_\sigma))_{\sigma\in\Sigma},$$

are also bounded.

The proof is easy and we left it to the reader.

4. Compact immersion of the Sobolev-like cone

In this section we present our main result. For $(R_n)_{n\in\mathbb{N}} \subseteq \mathscr{H}^1_{V,+}$, we denote by $(\nu_i^{(n)})_{i\in\mathbb{N}}$ and $(\psi_i^{(n)})_{i\in\mathbb{N}}$ the corresponding sequences of eigenvalues and eigenfunctions of R_n, for each $n \in \mathbb{N}$.

Theorem 4.1. *Let $m \in [d/(d+2), 1)$. Let V be a potential verifying* (H1)*,* (H2)*,* (H3) *and* (V_1)*. Let $(R_n)_{n\in\mathbb{N}} \subseteq \mathscr{H}^1_{V,+}$ be a sequence such that*

$$U_\infty = \sup_{n\in\mathbb{N}} \langle\langle R_n \rangle\rangle_V < \infty. \tag{4.1}$$

Up to a subsequence, $(R_n)_{n\in\mathbb{N}}$ converges in norm $\|\cdot\|_1$, to some $\overline{R} \in \mathscr{H}^1_{V,+}$.

We divide the proof of Theorem 4.1 in a sequence of Lemmas.

Lemma 4.1. *Under the conditions of Theorem 4.1, the sequence $(\|R_n\|_1)_{n\in\mathbb{N}}$ is bounded and*

$$\sup_{n\in\mathbb{N}} \sum_{i=1}^{\infty} (\nu_i^{(n)})^m < \infty, \tag{4.2}$$

Proof. By part iii) of Proposition 2.2 and (4.1), there exists $C > 0$ such that

$$\|R_n\|_1 = \sum_{i=1}^{\infty} \nu_i^{(n)} \leq C\langle\langle R_n \rangle\rangle_V \leq CU_\infty, \quad \text{for all } n \in \mathbb{N}, \tag{4.3}$$

so that $(\|R_n\|_1)_n$ is bounded. To get (4.2) we consider β as in (3.7), that is generated by F given in (3.4). Using (3.8) we have that

$$(1-m)^{m-1} m^{-m} \sum_{i=1}^{\infty} (\nu_i^{(n)})^m \leq \mathscr{K}(R_n) + \mathscr{P}_V(R_n) + \operatorname{Tr}[F(-\Delta + V)],$$

whence,

$$\sup_{n\in\mathbb{N}} \sum_{i=1}^{\infty} (\nu_i^{(n)})^m < (1-m)^{1-m} m^m \cdot (U_\infty + \operatorname{Tr}[F(-\Delta+V)]) < \infty.$$

□

Lemma 4.2. *Under the conditions of Theorem 4.1, for each $i \in \mathbb{N}$, up to a subsequence, there exists $\overline{\nu_i} \in \mathbb{R}^+_*$ such that*

$$\lim_{n\to\infty} \nu_i^{(n)} = \overline{\nu_i}. \tag{4.4}$$

Moreover, for each $i \in \mathbb{N}$, up to a subsequence, there exists $\overline{\psi_i} \in H^1(\Omega)$ such that

$$\lim_{n\to\infty} \psi_i^{(n)} = \overline{\psi_i}, \quad \text{in } L^2(\Omega). \tag{4.5}$$

Proof. Let $i \in \mathbb{N}$. By (4.3) we have that

$$\nu_i^{(n)} < CU_\infty, \quad \text{for all } n \in \mathbb{N}, \tag{4.6}$$

so that $(\nu_i^{(n)})_{n\in\mathbb{N}}$ is a bounded sequence and, therefore, there exists $\overline{\nu_i} \in \mathbb{R}$ such that, up to subsequence,

$$\lim_{n\to\infty} \nu_i^{(n)} = \overline{\nu_i}, \quad \text{for all } i \in \mathbb{N}. \tag{4.7}$$

Since $(\nu_i^{(n)})_{n\in\mathbb{N}} \subseteq \mathbb{R}_*^+$, it follows that

$$\overline{\nu_i} \geq 0, \quad \text{for all } i \in \mathbb{N}. \tag{4.8}$$

Now let's prove (4.5). For each $i, n \in \mathbb{N}$, we write

$$E_i^{(n)} = \|\psi_i^{(n)}\|_V^2 = \int_\Omega |\nabla \psi_i^{(n)}(x)|^2\, dx + \int_\Omega V(x)|\psi_i^{(n)}(x)|^2\, dx. \tag{4.9}$$

Since $(\nu_i^{(n)})_{n\in\mathbb{N}}$ is bounded, we have by (4.1) that $(E_i^{(n)})_{n\in\mathbb{N}}$ is bounded in $H_V(\Omega)$. Then, by Proposition 2.1 there exists $\overline{\psi_i} \in L^2(\Omega)$ such that, up to subsequence, it verifies

$$\lim_{n\to\infty} \psi_i^{(n)} = \overline{\psi_i}, \quad \text{in } L^2(\Omega).$$

Using [2, Prop. IX.3] we prove that $\overline{\psi_i} \in H^1(\Omega)$ and then (4.1) implies that $\overline{\psi_i} \in H_V(\Omega)$. □

Lemma 4.3. *Under the conditions of Theorem 4.1, up to a subsequence, we have that*

$$\lim_{n\to\infty} \sum_{i=1}^{\infty} (\nu_i^{(n)})^m = \sum_{i=1}^{\infty} (\overline{\nu_i})^m. \tag{4.10}$$

Proof. We have to prove that given $\varepsilon > 0$ there exists $N \in \mathbb{N}$ such that

$$\sum_{i=N}^{\infty} (\nu_i^{(n)})^m \leq \varepsilon, \quad \text{for all } n \in \mathbb{N}. \tag{4.11}$$

Let's recall that for $p \in (0,1)$ and $q \in (-\infty, 0)$ such that $\frac{1}{p} + \frac{1}{q} = 1$, the reverse Hölder inequality holds

$$\sum_{i=1}^{\infty} \xi_i \eta_i \geq \left(\sum_{i=1}^{\infty} \xi_i^p\right)^{1/p} \left(\sum_{i=1}^{\infty} \eta_i^q\right)^{1/q},$$

for all $(\xi_i)_{i\in\mathbb{N}} \in \ell^p(\mathbb{R}^+)$ and $(\eta_i)_{i\in\mathbb{N}} \in \ell^q(\mathbb{R}^+)$. Therefore, by choosing $p = m$ and $q = -\gamma$ we have that

$$\left(\sum_{i=N}^{\infty} |\nu_i^{(n)}|^m\right)^{1/m} \leq U_\infty \left(\sum_{i=N}^{\infty} |E_i^{(n)}|^{-\gamma}\right)^{1/\gamma}, \quad \text{for all } N \in \mathbb{N}. \tag{4.12}$$

Let's notice that

$$E_i^{(n)} = \sum_{k=1}^{\infty} |(\phi_{V,k}, \psi_i^{(n)})|^2 \lambda_{V,k}, \tag{4.13}$$

$$\sum_{k=1}^{\infty} |(\phi_{V,k}, \psi_i^{(n)})|^2 = 1. \tag{4.14}$$

Since $\gamma \geq d/2$, then $(0,\infty) \ni s \to s^{-\gamma} \in \mathbb{R}$ is convex. Therefore, by (4.12) and (4.13), we get

$$(E_i^{(n)})^{-\gamma} \leq \sum_{k=1}^{\infty} |(\phi_{V,k}, \psi_i^{(n)})|^2 (\lambda_{V,k})^{-\gamma}, \quad \text{for all } i \in \mathbb{N} \text{ and } n \in \mathbb{N}. \tag{4.15}$$

Now, since $0 < \lambda_{V,1} < \lambda_{V,2} \leq \ldots \leq \lambda_{V,k} \leq \ldots$ and $\sum_{k=1}^{\infty} (\lambda_{V,k})^{-\gamma} < \infty$, by (4.13), (4.14) and (4.15), we can choose $N, M \in \mathbb{N}$ such that

$$\begin{aligned}
\sum_{i=N}^{\infty} (E_i^{(n)})^{-\gamma} &= \sum_{k=1}^{M-1} \sum_{i=N}^{\infty} |(\phi_{V,k}, \psi_i^{(n)})|^2 (\lambda_{V,k})^{-\gamma} + \sum_{k=M}^{\infty} \sum_{i=N}^{\infty} |(\phi_{V,k}, \psi_i^{(n)})|^2 (\lambda_{V,k})^{-\gamma} \\
&\leq \frac{M-1}{(\lambda_{V,1})^{\gamma}} \sum_{i=N}^{\infty} |(\phi_{V,k}, \psi_i^{(n)})|^2 + \sum_{k=M}^{\infty} \sum_{i=N}^{\infty} |(\phi_{V,k}, \psi_i^{(n)})|^2 (\lambda_{V,k})^{-\gamma} \\
&\leq \left(\frac{\varepsilon}{U_\infty} \right)^{\gamma}.
\end{aligned} \tag{4.16}$$

We conclude by combining (4.12) and (4.16). □

Lemma 4.4. *Under the conditions of Theorem 4.1, for all $m' \in [m,1]$, up to a subsequence, it verifies that*

$$\lim_{n\to\infty} \sum_{i=1}^{\infty} (\nu_i^{(n)})^{m'} = \sum_{i=1}^{\infty} (\overline{\nu_i})^{m'}. \tag{4.17}$$

Proof. Let $m' \in [m,1]$. We have to prove that given $\varepsilon > 0$ there exists $N \in \mathbb{N}$ such that

$$\sum_{i=N}^{\infty} (\nu_i^{(n)})^{m'} < \varepsilon. \tag{4.18}$$

Since $(\nu_i^{(n)})_{i\in\mathbb{N}}$ is ordered, it verifies that

$$(\nu_N^{(n)})^{m'-m} \geq (\nu_j^{(n)})^{m'-m}, \quad \text{for all } N \leq j, \tag{4.19}$$

$$\sum_{i=N}^{\infty} (\nu_i^{(n)})^m = \sum_{j\in A_N} (\nu_i^{(n)})^m,$$

where $A_N = \{i \in \mathbb{N} / \, i \geq N \wedge \nu_i^{(n)} \neq 0\}$. Then, by (4.11) and (4.19), we conclude:

$$\sum_{i=N}^{\infty} (\nu_i^{(n)})^{m'} = \sum_{i=N}^{\infty} (\nu_i^{(n)})^{m'-m} (\nu_i^{(n)})^m \leq (\nu_N^{(n)})^{m'-m} \sum_{i=N}^{\infty} (\nu_i^{(n)})^m \leq \varepsilon. \tag{4.20}$$

□

We use Lemmas 4.1 - 4.4 to prove Theorem 4.1.

Proof of Theorem 4.1. Let's prove that the operator $\overline{R} : L^2(\Omega) \to L^2(\Omega)$ given by

$$\overline{R}\eta = \sum_{i=1}^{\infty} \overline{\nu_i}(\overline{\psi_i}, \eta)\overline{\psi_i}.$$

belongs to $\mathscr{H}^1_{V,+}$ and

$$\lim_{n\to\infty} \|R_n - \overline{R}\|_1 = 0. \tag{4.21}$$

It's not dificult to verify that $\overline{R}$ is a self-adjoint, positive and trace-class operator. By Fatou's Lemma, we have that

$$\mathscr{K}(\overline{R}) \le \liminf_{n\to\infty} \int_\Omega \sum_{i=1}^{\infty} \nu_i^{(n)} |\nabla \psi_i^{(n)}(x)|^2 \, dx < U_\infty, \tag{4.22}$$

$$\mathscr{P}(\overline{R}) \le \liminf_{n\to\infty} \int_\Omega \sum_{i=1}^{\infty} \nu_i^{(n)} |\psi_i^{(n)}(x)|^2 V(x) \, dx < U_\infty, \tag{4.23}$$

so that $\overline{R} \in \mathscr{H}^1_{V,+}$. For each $N \in \mathbb{N}$ we consider the orthogonal projections $P_N^n : L^2(\Omega) \to F_N^n$ and $Q_N^n = Id - P_N^n$ given by

$$P_N^n(\eta) = \sum_{i=1}^{N} (\eta, \psi_i^{(n)})\psi_i^{(n)},$$

where

$$F_N^n = \operatorname{span}\{\psi_i^{(n)} : i = 1, \dots, N-1\}.$$

Q_N^n is the orthogonal projection on $(F_N^n)^\perp$. We also consider $P_N : L^2(\Omega) \to F_N$ and $Q_N = Id - P_N$ given by

$$P_N(\eta) = \sum_{i=1}^{N} (\eta, \overline{\psi_i})\overline{\psi_i},$$

where

$$F_N = \operatorname{span}\{\overline{\psi_i} : i = 1, \dots, N-1\}.$$

Now given $\varepsilon > 0$, we can choose $N \in \mathbb{N}$ so that

$$\|R_n - \overline{R}\|_1 \le \|(R_n - \overline{R})P_N\|_1 + \|R_n Q_N^n\|_1 + \|\overline{R} Q_N\|_1 + \|R_n(Q_N - Q_N^n)\|_1 \le \varepsilon,$$

for $n \in \mathbb{N}$ large enough. ☐

5. Applications

In this section we apply Theorem 4.1 to minimize two types of non-linear free energy functional. First we consider a generic free energy functional.

Theorem 5.1. *Let V be a potential verifying* (H1)*,* (H2)*,* (H3) *and* (V_1)*. Assume that $-\Delta+V$ is $\varepsilon-$coercive, $\varepsilon\in(0,1]$. Let $\lambda\leq\lambda_{V,1}^{(1-\varepsilon)}$. Let β be a lower semicontinuous entropy seed generated by $F\in\mathscr{C}_0^{\varepsilon/2}\cap\mathscr{C}_{V-\lambda}^{1-\varepsilon}$. Then there exists a unique $\overline{R}\in\mathscr{H}_{V,+}^1$ such that*

$$\mathscr{F}_{V,\beta}^{\lambda}(\overline{R})=\inf_{R\in\mathscr{H}_{V,+}^1}\mathscr{F}_{V,\beta}^{\lambda}(R). \tag{5.1}$$

Proof. By Proposition 3.3 we have that

$$\mathscr{F}_{V,\beta}^{\lambda}(R)\geq-\mathrm{Tr}\left[F(-(1-\varepsilon)\Delta+V-\lambda)\right],\quad \text{for all } R\in\mathscr{H}_{V,+}^1,$$

i.e. $\mathscr{F}_{V,\beta}^{\lambda}$ is bounded from below. Then we choose $(R_n)_{n\in\mathbb{N}}\subseteq\mathscr{H}_{V,+}^1$ such that

$$\lim_{n\to\infty}\mathscr{F}_{V,\beta}^{\lambda}(R_n)=\inf_{R\in\mathscr{H}_{V,+}^1}\mathscr{F}_{V,\beta}^{\lambda}(R), \tag{5.2}$$

Since $(\mathscr{F}_{V,\beta}^{\lambda}(R_n))_{n\in\mathbb{N}}\subseteq\mathbb{R}$ is bounded, Proposition 3.2 implies that the sequences $(\|R_n\|_1)_{n\in\mathbb{N}}$, $(\mathscr{K}(R_n))_{n\in\mathbb{N}}$, $(\mathscr{P}_V(R_n))_{n\in\mathbb{N}}$ and $(\mathscr{E}_\beta(R_n))_{n\in\mathbb{N}}$ are bounded. In particular, there exists $U_\infty>0$ such that

$$U_\infty=\sup_{n\in\mathbb{N}}\{\mathscr{K}(R_n)+\mathscr{P}_V(R_n)\}.$$

Then, by Theorem 3.1, there exists $\overline{R}\in\mathscr{H}_{V,+}^1$ such that, up to subsequences, verifies

$$\lim_{n\to\infty}\|R_n-\overline{R}\|_1=0, \tag{5.3}$$

$$\mathscr{K}(\overline{R})\leq\liminf_{n\to\infty}K(R_n), \tag{5.4}$$

$$\mathscr{P}_V(\overline{R})\leq\liminf_{n\to\infty}\mathscr{P}_V(R_n), \tag{5.5}$$

by (4.22) and (4.23). On other hand, putting $A=\sup_{n\in\mathbb{N}}(\mathscr{E}_\beta(R_n))$, let's consider on the convex set

$$A_+=\left\{a=(a_i)_{i\in\mathbb{N}}\in\ell^1(\mathbb{R}^+)\,:\,\sum_{i=1}^{\infty}\beta(a_i)<A\right\},$$

the application $D:A_+\to\mathbb{R}$ given by

$$D(a)=\sum_{i=1}^{\infty}\beta(a_i).$$

D is convex and lower semicontinuous so that

$$\mathscr{E}_\beta(\overline{R})\leq\liminf_{n\to\infty}\mathscr{E}_\beta(R_n). \tag{5.6}$$

Therefore, by (5.3), (5.4), (5.5) and (5.6), it verifies

$$\begin{aligned}\mathscr{F}_{V,\beta}^{\lambda}(\overline{R}) &= \mathscr{E}_\beta(\overline{R})+\mathscr{K}(\overline{R})+\mathscr{P}_V(\overline{R})-\lambda\|\overline{R}\|_1\\ &\leq \liminf_{n\to\infty}[\mathscr{E}_\beta(R_n)+\mathscr{K}(R_n)+\mathscr{P}_V(R_n)-\lambda\|R_n\|_1]\\ &= \inf_{R\in\mathscr{H}_{V,+}^1}\mathscr{F}_{V,\beta}^{\lambda}(R), \end{aligned}\tag{5.7}$$

so that $\overline{R}$ verifies (5.1). Using mixed states, we stablish the uniqueness of $\overline{R}$ as it's done in the proof of [5, Th. 4.1]. □

Remark 5.1. Since formally $\mathscr{F}^{\lambda}_{V,\beta}(R) = \mathrm{Tr}\left((-\Delta + V - \lambda)R + \beta(R)\right)$, as a critical point of $\mathscr{F}^{\lambda}_{V,\beta}$ in Theorem 5.1, $\overline{R}$ should have the form

$$\overline{R} = (\beta')^{-1}(-(-\Delta + V) + \lambda).$$

Now let's consider a free energy functional involving a non-linear but local function of the density. Let g be a real function. Formally, for $R \in \mathscr{H}^1_{V,+}$, we write

$$\mathscr{G}(R) = \int_{\Omega} g(\rho_R(x))dx,$$
$$\mathscr{F}^{\lambda,g}_{V,\beta}(R) = \mathscr{F}^{\lambda}_{V,\beta}(R) + \mathscr{G}(R).$$

Theorem 5.2. *Let V be a potential verifying* (H1)*,* (H2)*,* (H3) *and* (V_1)*. Assume that $-\Delta + V$ is $\varepsilon-$coercive, $\varepsilon \in (0,1]$. Let $\lambda \leq \lambda^{(1-\varepsilon)}_{V,1}$. Let β be a lower semicontinuous seed entropy generated by $F \in \mathscr{C}^{\varepsilon/2}_0 \cap \mathscr{C}^{1-\varepsilon}_{V-\lambda}$ and $g \in C([0,+\infty))$ such that*

$$C_1 \leq g(s) \leq C_2 s^q, \quad \text{for all } s \geq 0, \tag{5.8}$$

for some constants $C_1, C_2 > 0$ and some $q \in [1, d/d-2]$. Then there exists a unique $\overline{R} \in \mathscr{H}^1_{V,+}$ such that

$$\mathscr{F}^{\lambda,g}_{V,\beta}(\overline{R}) = \inf_{R \in \mathscr{H}^1_{V,+}} \mathscr{F}^{\lambda,g}_{V,\beta}(R). \tag{5.9}$$

Proof. We proceed as in the proof of Theorem 5.1 and observe that for a sequence $(R_n)_{n\in\mathbb{N}} \subseteq \mathscr{H}^1_{V,+}$ minimizing $\mathscr{F}^{\lambda,g}_{V,\beta}$, it holds by (5.8) and Proposition 2.3, that

$$\mathscr{G}(\overline{R}) \leq \liminf_{n\to\infty} \mathscr{G}(R_n), \tag{5.10}$$

where, up to subsequence, $\lim\limits_{n\to\infty} \|R_n - \overline{R}\|_1 = 0$, with $\overline{R} \in \mathscr{H}^1_{V,+}$. Then, by (5.3), (5.4), (5.5), (5.6) and (5.10) we have that

$$\mathscr{F}^{\lambda,g}_{V,\beta}(\overline{R}) = \inf_{R \in \mathscr{H}^1_{V,+}} \mathscr{F}^{\lambda,g}_{V,\beta}(R).$$

□

Remark 5.2. Formally, the minimizer of $\mathscr{F}^{\lambda,g}_{V,\beta}$ is a fixed point of the aplication $Y : \mathscr{H}^1_{V,+} \to \mathscr{H}^1_{V,+}$ given by

$$Y(R) = (\beta')^{-1}(-(-\Delta + V) + \lambda - g' \circ \rho_R).$$

6. Conclusions

Given an smooth unbounded domain $\Omega \subseteq \mathbb{R}^d$, $d \geq 3$, we have extended the results obtained in [5] (where Ω was assumed to be bounded). Our setting could physically correspond to an external potential having a singularity, as it's the case of some potentials generated by doping charged impurities in semiconductors.

A positive self-adjoint trace-class operator R belongs to the Sobolev-like cone $\mathscr{H}^1_{V,+}$ if its Hilbertian eigenbase (for $L^2(\Omega)$) is included in the normed space $H_V(\Omega)$ and has finite energy

$$\langle\langle R \rangle\rangle_V = \sum_{i=1}^{\infty} \nu_{i,R} \|\psi_{i,R}\|_V^2,$$

where $(\nu_{i,R})_{i\in\mathbb{N}}$ is the sequence of eigenvalues of R. Here

$$H_V(\Omega) = \left\{ u \in H_0^1(\Omega) : \|u\|_V^2 = \int_\Omega (|\nabla u(x)|^2 + |u(x)|^2 V(x))\, dx < \infty \right\},$$

where the potential V is non-negative on Ω and blowing up at infinity. For us it was a key the well known property that $H_V(\Omega)$ immerses compactly in $L^q(\Omega)$, $q \in [2, 2^*]$.

We proved that a sequence in $\mathscr{H}^1_{V,+}$, bounded in energy $\langle\langle \cdot \rangle\rangle_V$, has a subsequence that converges in trace norm; this is analogous to the classical Sobolev immersion $\mathrm{H}^1(\Omega) \subseteq \mathrm{L}^2(\Omega)$. By proving the lower boundedness of non-linear free energy functionals more general than the one given by

$$\mathscr{F}_{V,\beta}(R) = \mathrm{Tr}\left((-\Delta + V)R + \beta(R)\right), \quad R \in \mathscr{H}^1_{V,+},$$

we established Lieb-Thirring type inequalities as well some Gagliardo-Nirenberg type interpolation inequalities. Then our compactness result was applied to minimize those non-linear free energy functionals.

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