WELL-POSEDNESS FOR A HIGHER ORDER NONLINEAR SCHRÖDINGER EQUATION IN SOBOLEV SPACES OF NEGATIVE INDICES.

XAVIER CARVAJAL

ABSTRACT. We prove that, the initial value problem associated to

$$\partial_t u + i\alpha\partial_x^2 u + \beta\partial_x^3 u + i\gamma|u|^2 u = 0, \quad x, t \in \mathbb{R},$$

is locally well-posed in $\mathbf{H^s}$ for $s > -1/4$.

1. INTRODUCTION

In this work, we study a particular case of the following initial value problem (IVP)

$$\begin{aligned} \partial_t u + i\alpha\partial_x^2 u + \beta\partial_x^3 u + F(u) &= 0, \quad x, t \in \mathbb{R}, \\ u(x,0) &= u_0(x) \end{aligned} \tag{1}$$

where u is a complex valued function, $F(u) = i\gamma|u|^2 u + \delta|u|^2\partial_x u + \epsilon u^2 \partial_x \overline{u}$, $\gamma, \delta, \epsilon \in \mathbb{C}$ and $\alpha, \beta \in \mathbb{R}$ are constants.

A. Hasegawa and Y. Kodama [7, 12], proposed (1) as a model for propagation of pulse in optical fiber. We will study the IVP (1) in Sobolev space $\mathbf{H^s}(\mathbb{R})$ under the condition $\delta = \epsilon = 0$, $\beta \neq 0$ (see case **iv)** in Teorema 1 below). Our definition of local well-posedness includes: existence, uniqueness, persistence and continuous dependence of solution on given data (i.e. continuity of application $u_0 \mapsto u(t)$ from X to $\mathcal{C}([-T,T];X)$).

If $T < \infty$ we say that the IVP is locally well-posed in X. If some hypothesis in the definition of local well-posedness fails, we say that the IVP is ill-posed.

Particular cases of (1) are the followings:

• Cubic nonlinear Schrödinger equation (NLS), ($\alpha = \mp 1$, $\beta = 0$, $\gamma = -1$, $\delta = \epsilon = 0$).

$$iu_t \pm u_{xx} + |u|^2 u = 0, \quad x, t \in \mathbb{R}. \tag{2}$$

Best known local result for the IVP associated to (2) is in $\mathbf{H^s}(\mathbb{R})$, $s \geq 0$, obtained by Tsutsumi [21].

• Nonlinear Schrödinger equation with derivative ($\alpha = -1$, $\beta = 0$, $\gamma = 0$, $\delta = 2\epsilon$).

$$iu_t + u_{xx} + i\lambda(|u|^2 u)_x = 0, \quad x, t \in \mathbb{R}. \tag{3}$$

Best known result for the IVP associated to (3) is in $\mathbf{H^s}(\mathbb{R})$, $s \geq 1/2$, obtained by Takaoka [19].

• Complex modified Korteweg-de Vries (mKdV) equation ($\alpha = 0$, $\beta = 1$, $\gamma = 0$, $\delta = 1$, $\epsilon = 0$).

$$u_t + u_{xxx} + |u|^2 u_x = 0, \quad x, t \in \mathbb{R}. \tag{4}$$

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If u is real, (4) is the usual mKdV equation and Kenig et al. [9], proved the IVP associated to it is locally well-posed in $\mathbf{H^s}(\mathbb{R})$, $s \geq 1/4$.

Laurey [15, 14] proved that the IVP associated to (1) is locally well-posed in $\mathbf{H^s}(\mathbb{R})$, $s > 3/4$.

Staffilani [17] improved this result by proving the IVP associated to (1) is locally well-posed in $\mathbf{H^s}(\mathbb{R})$, $s \geq 1/4$.

When α, β are functions of t, we proved in [1, 2] local well-posedness in $\mathbf{H^s}(\mathbb{R})$, $s \geq 1/4$. Also we studied in [1, 4] the unique continuation property for the solution of (1). Regarding the ill-posedness of the IVP (1), we proved in [3] the following theorem.

Theorem 1. *The mapping data-solution $u_0 \mapsto u(t)$ for the IVP (1) is not $\mathcal{C}^3$ at origin in the following cases:*
i) $\beta = 0$, $\alpha \neq 0$, $\delta = \epsilon = 0$, $\gamma \neq 0$ *for* $s < 0$.
ii) $\beta = 0$, $\alpha \neq 0$, $\delta \neq 0$ *or* $\epsilon \neq 0$ *for* $s < 1/2$.
iii) $\beta \neq 0$, $\delta \neq 0$ *or* $\epsilon \neq 0$ *for* $s < 1/4$.
iv) $\beta \neq 0$, $\delta = \epsilon = 0$, $\gamma \neq 0$ *for* $s < -1/4$.

In this work, considering the case **iv)** in Theorem 1, we prove the following theorem.

Theorem 2. *The IVP associated to* **iv)**,
$$\partial_t u + i\alpha\partial_x^2 u + \beta\partial_x^3 u + i\gamma|u|^2 u = 0, \quad x, t \in \mathbb{R}, \tag{5}$$
is locally well-posed in $\mathbf{H^s}(\mathbb{R})$, $s > -1/4$.

The following trilinear estimate will be fundamental in the proof of Theorem 2

Theorem 3. *Let* $-1/4 < s \leq 0$, $7/12 < b < 11/12$, *then we have*
$$\|uv\overline{w}\|_{X^{s,b-1}} \leq C\|u\|_{X^{s,b}}\|v\|_{X^{s,b}}\|w\|_{X^{s,b}}, \tag{6}$$
where
$$\|u\|_{X^{s,b}} = \| <\xi>^s <\tau - \phi(\xi)>^b \hat{u}\|_{\mathbf{L^2_\xi L^2_\tau}},$$
$<\xi> = 1 + |\xi|$, $\phi(\xi) = \alpha\xi^2 + \beta\xi^3$.

Theorem 4. *The trilinear estimate (6) fails if* $s < -1/4$ *and* $b \in \mathbb{R}$.

Remark 1. *1) As the equation (1) preserves* $\mathbf{L^2}$ *norm, the Theorem 2 permits to obtain global existence in* $\mathbf{L^2}$.

2) From Lemma 3 we note that the value $7/12+$ *is the best possible for the value very near to* $\rho = 1/4$, *in the trilinear estimate (6).*

3) The trilinear estimate is valid for all $s > 0$, *because it follows by combing the fact that* $<\xi>^s \leq <\xi - (\xi_2 - \xi_1)>^s <\xi_2>^s <\xi_1>^s$ *and the estimate (6) for* $s = 0$.

4) We will use the notation $\|u\|_{\{s,b\}} := \|u\|_{X^{s,b}}$.

5) When $\alpha = 0, \beta = 1$, *we have* $-3/4+$ *bilinear estimate* [10],
$$\|(uv)_x\|_{\{-3/4+,-1/2+\}} \leq C\|u\|_{\{-3/4+,1/2+\}}\|v\|_{\{-3/4+,1/2+\}}.$$
Also we have the 1/4 trilinear estimate [20],
$$\|(uvw)_x\|_{\{1/4,-1/2+\}} \leq C\|u\|_{\{1/4,1/2+\}}\|v\|_{\{1/4,1/2+\}}\|w\|_{\{1/4,1/2+\}}.$$

2. Proof of Theorem 4.

As in [10] consider the set

$$B := \{(\xi,\tau); N \le \xi \le N + N^{-1/2}, |\tau - \phi(\xi)| \le 1\},$$

where $\phi(\xi) = \alpha\xi^2 + \beta\xi^3$. We have $|B| \sim N^{-1/2}$. Let us consider $\hat{v} = \chi_B$, it is not difficult to see that $\|v\|_{\{s,b\}} \le N^s |B|^{1/2}$. Moreover

$$\mathcal{F}(|v|^2\overline{v}) := \chi_B * \chi_B * \chi_{-B} \gtrsim \frac{1}{N}\chi_A,$$

where A is a rectangle contained in B such that $|A| \sim N^{-1/2}$.
Therefore

$$\begin{aligned}\| \, |v|^2\overline{v}\|_{\{s,b-1\}} =& \| < \xi >^s < \tau - \phi(\xi) >^{b-1} \mathcal{F}(|v|^2\overline{v})\|_{\mathbf{L^2_\xi L^2_\tau}} \\ \gtrsim& N^s \frac{1}{N} N^{-1/4} = N^{s-5/4}.\end{aligned}$$

As a consequence, for large N the trilinear estimate fails if $3(s-1/4) < s - 5/4$, i.e. if $s < -1/4$. □

3. Proof of Theorem 3.

To prove Theorem 3, we need the following results from elementary calculus.

Lemma 1. (1) *If $b > 1/2$, $a_1, a_2 \in \mathbb{R}$*

$$\int_{\mathbb{R}} \frac{dx}{< x - a_1 >^{2b} < x - a_2 >^{2b}} \sim \frac{1}{< a_1 - a_2 >^{2b}}. \tag{7}$$

(2) *If $0 < c_1, c_2 < 1$, $c_1 + c_2 > 1$, $a_1 \ne a_2$, then*

$$\int_{\mathbb{R}} \frac{dx}{|x - a_1|^{c_1} |x - a_2|^{c_2}} \lesssim \frac{1}{|a_1 - a_2|^{(c_1+c_2-1)}}. \tag{8}$$

(3) *Let $a \in \mathbb{R}$, $c_1 \le c_2$, then*

$$\frac{|x|^{c_1}}{< ax >^{c_2}} \le \frac{C(c_1,c_2)}{a^{c_1}}, \tag{9}$$

where $C(c_1,c_2)$ is a constant independent of x.

(4) *Let $a, \eta \in \mathbb{R}$, $b > 1/2$, then*

$$\int_{\mathbb{R}} \frac{dx}{< a(x^2 - \eta^2) >^{2b}} \lesssim \frac{1}{|a\eta|}. \tag{10}$$

Let $f(\xi,\tau) = < \xi >^s < \tau - \xi^3 >^b \hat{u}$, $g(\xi,\tau) = < \xi >^s < \tau - \xi^3 >^b \hat{v}$, $h(\xi,\tau) = < \xi >^s < \tau - \xi^3 >^b \hat{w}$, $\eta = (\xi,\tau)$, $x = (\xi_1,\tau_1)$, $y = (\xi_2,\tau_2)$.
We have

$$\begin{aligned}\|uv\overline{w}\|_{\{s,b-1\}} =& \| \int_{\mathbb{R}^4} f(\eta + x - y) g(y) \overline{h}(x) K(\eta,x,y) dx dy \|_{\mathbf{L^2_\eta}} \\ \le& \|K(\eta,x,y)\|_{\mathbf{L^\infty_\eta L^2_{x,y}}} \|f\|_{\mathbf{L^2}} \|g\|_{\mathbf{L^2}} \|h\|_{\mathbf{L^2}},\end{aligned}$$

where $r(\xi,\tau) = <\xi>^{2\rho} <\tau-\phi(\xi)>^{2(1-b)}$, $\rho=-s$ and

$$K(\eta,x,y) = \frac{<\xi+\xi_1-\xi_2>^{2\rho}<\xi_2>^{2\rho}<\xi_1>^{2\rho}}{r(\xi,\tau)\ <\tau_1-\phi(\xi_1)>^{2b}<\tau_2-\phi(\xi_2)>^{2b}<\tau+\tau_1-\tau_2-\phi(\xi+\xi_1-\xi_2)>^{2b}}.$$

Using (7) we obtain

$$I(\xi,\tau) := \|K(\xi,\tau)\|^2_{\mathbf{L^2_{x,y}}} \sim \frac{1}{r(\xi,\tau)} \int_{\mathbb{R}^2} \frac{G_\rho(\xi,\xi_1,\xi_2)\ d\xi_1 d\xi_2}{<\tau-\phi(\xi+\xi_1-\xi_2)-\phi(\xi_2)+\phi(\xi_1)>^{2b}},$$

where $G_\rho(\xi,\xi_1,\xi_2) := <\xi+\xi_1-\xi_2>^{2\rho}<\xi_1>^{2\rho}<\xi_2>^{2\rho}$.

For clarity in exposition we consider the case $\alpha=0$, $\beta=1$, i.e. $\phi(\xi)=\xi^3$. With this consideration we have

$$I(\xi,\tau) := \frac{1}{<\xi>^{2\rho}<\tau-\xi^3>^{2(1-b)}} \int_{\mathbb{R}^2} \frac{G_\rho(\xi,-\xi_1,-\xi_2) d\xi_1 d\xi_2}{<\tau-\xi^3+g>^{2b}},$$

where $g=g(\xi,\xi_1,\xi_2)=3(\xi_1-\xi_2)(\xi-\xi_1)(\xi+\xi_2)$.

Supposing $y=\tau-\xi^3$, to get Theorem 3 it is enough to prove

Lemma 2. *Let $0<\rho<1/4$, $7/12<b<11/12$. Then*

$$I(\xi,y) := \frac{1}{<\xi>^{2\rho}<y>^{2(1-b)}} \int_{\mathbb{R}^2} \frac{G_\rho(\xi,-\xi_1,-\xi_2) d\xi_1 d\xi_2}{<y+g(\xi,\xi_1,\xi_2)>^{2b}} \le C(\rho,b) < \infty,$$

where $C(\rho,b)$ is a constant independent of ξ and y.

To prove Lemma 2 we need to prove the following lemmas.

Lemma 3. *Let $\rho<1/4$, then we have*

$$I(0,0) = \int_{\mathbb{R}^2} \frac{G_\rho(0,-\xi_1,-\xi_2) d\xi_1 d\xi_2}{< g(0,\xi_1,\xi_2) >^{2b}} = \begin{cases} C(\rho,b)<\infty, & \text{if } \rho+1/3<b \\ \infty, & \text{if } \rho+1/3\ge b, \end{cases}$$

where $C(\rho,b)$ is a constant.

Lemma 4. *Let $\rho<1/4$, $b>7/12$, then*

$$I(\xi,0) = \frac{1}{<\xi>^{2\rho}} \int_{\mathbb{R}^2} \frac{G_\rho(\xi,-\xi_1,-\xi_2) d\xi_1 d\xi_2}{<g(\xi,\xi_1,\xi_2)>^{2b}} \le C(\rho,b),$$

where $C(\rho,b)$ is a constant independent of ξ.

In the definition of $I(\xi,y)$ if we make the change of variables $\xi-\xi_1 := \xi\xi_1$, $\xi+\xi_2 := \xi\xi_2$ and $y=\xi^3 z$, then $I(\xi,y)$ becomes

$$I(\xi,z) = p(\xi,z) \int_{\mathbb{R}^2} \frac{H_\rho(\xi,\xi_1,\xi_2) d\xi_1 d\xi_2}{< \xi^3(z+F(\xi_1,\xi_2)) >^{2b}}, \tag{11}$$

where $p(\xi,z)=\xi^2/<\xi^3 z>^{2(1-b)}<\xi>^{2\rho}$, $F(\xi_1,\xi_2)=(2-(\xi_1+\xi_2))\xi_1\xi_2$ and

$$H_\rho(\xi,\xi_1,\xi_2) = <\xi(1-(\xi_1+\xi_2))>^{2\rho}<\xi(1-\xi_1)>^{2\rho}<\xi(1-\xi_2)>^{2\rho}.$$

From here onwards we will suppose $z>0$.

Proof of Lemma 3.

By symmetry it is enough to prove that the following integrals

$$I_1(0,0) := \int_0^\infty \int_0^\infty \frac{G_\rho(0,-\xi_1,-\xi_2) d\xi_1 d\xi_2}{< g(0,\xi_1,\xi_2) >^{2b}}, \quad I_2(0,0) := \int_0^\infty \int_0^\infty \frac{G_\rho(0,-\xi_1,\xi_2) d\xi_1 d\xi_2}{< g(0,\xi_1,-\xi_2) >^{2b}}$$

are finite. We will prove $I_1(0,0)$ is finite, the same proof works to prove $I_2(0,0)$ is finite. Also, by symmetry we can suppose that $0 \leq \xi_2 \leq \xi_1$.

We have

$$\int_1^\infty d\xi_1 \int_0^{\xi_1} d\xi_2 \frac{G_\rho(0,-\xi_1,-\xi_2)}{< g(0,\xi_1,\xi_2) >^{2b}} = \int_1^\infty d\xi_1 \int_0^{\xi_1/2} d\xi_2 + \int_1^\infty d\xi_1 \int_{\xi_1/2}^{\xi_1} d\xi_2$$
$$= I_{1,1} + I_{1,2}.$$

As $0 \leq \xi_2 \leq \xi_1$, we have $G_\rho(0,-\xi_1,-\xi_2) \leq < \xi_1 >^{4\rho} < \xi_2 >^{2\rho}$. In $I_{1,1}$ we have $\xi_1/2 < \xi_1 - \xi_2 < \xi_1$, therefore if $b > \rho + 1/3$,

$$I_{1,1} \lesssim \int_1^\infty < \xi_1 >^{4\rho} d\xi_1 \int_0^{\xi_1/2} \frac{< \xi_2 >^{2\rho} d\xi_2}{< \xi_1^2 \xi_2 >^{2b}} \tag{12}$$

$$\lesssim \int_1^\infty < \xi_1 >^{4\rho} \Big(\frac{1}{\xi_1^2} + \frac{1}{\xi_1^{2+4\rho}} + \frac{1}{\xi_1^{2+4\rho}} \int_1^{3\xi_1^3/2} \frac{x^{2\rho} dx}{(1+x)^{2b}} \Big) d\xi_1 \tag{13}$$

$$= C(\rho, b) < \infty. \tag{14}$$

Analogously we can prove that $I_{1,1} = \infty$ if $b \leq \rho + 1/3$.

In $I_{1,2}$ we have $\xi_1/2 \leq \xi_2 \leq \xi_1$, so

$$I_{1,2} \lesssim \int_1^\infty < \xi_1 >^{4\rho} d\xi_1 \int_{\xi_1/2}^{\xi_1} \frac{< \xi_1 - \xi_2 >^{2\rho} d\xi_2}{< (\xi_1 - \xi_2)\xi_1^2 >^{2b}}$$
$$= \int_1^\infty < \xi_1 >^{4\rho} d\xi_1 \int_0^{\xi_1/2} \frac{< x >^{2\rho} dx}{< \xi_1^2 x >^{2b}}$$
$$= C(\rho, b), \quad b > \rho + 1/3.$$

□

To prove Lemmas 2 and 4, the following propositions will be useful.

Proposition 1. *Let $\rho \geq 0$, $b > 1/3 + 2\rho/3$, then we have*

$$J_2 = \xi^{2+4\rho} \int_{\mathbb{R}^2} \frac{d\xi_1 d\xi_2}{< \xi^3(z+F) >^{2b}} \leq C,$$

where C is a constant independent of ξ.

Proof: If $\xi_1 \leq 0$, $\xi_2 \leq 0$, then $|z+F| \geq |\xi_1 + \xi_2||\xi_1\xi_2|$. Therefore by Lemma 3 and by symmetry, it is enough to consider $\xi_1 \geq 0$. We have $|z+F| = |\xi_1||(\xi_2 + (\xi_1 - 2)/2)^2 - (\xi_1 - 2)^2/4 - z/\xi_1|$. Let $l^2 = (\xi_1 - 2)^2/4 + z/\xi_1$, $c(\rho) = (2+4\rho)/3$, then making change of variable $\eta = \xi_2 + (\xi_1 - 2)/2$ and using (8) and (9) we have

$$J_2 = \xi^{2+4\rho} \int_0^\infty d\xi_1 \int_{\mathbb{R}} \frac{d\eta}{< \xi^3 \xi_1 (\eta^2 - l^2) >^{2b}}$$
$$\lesssim \int_0^\infty d\xi_1 \int_{\mathbb{R}} \frac{l\, dx}{[|\xi_1| l^2 |x^2 - 1|]^{c(\rho)}}$$
$$\lesssim \int_0^\infty \frac{d\xi_1}{|\xi_1|^{c(\rho)} |\xi_1 - 2|^{(1+8\rho)/3}} \int_{\mathbb{R}} \frac{dx}{|x^2 - 1|^{c(\rho)}}$$
$$\lesssim C.$$

□

Proposition 2. *Let $|\xi| > 1$, $\rho < 1/4$, then*

$$J_1 = \xi^{2+4\rho} \int_0^\infty \xi_1^{4\rho} \int_{\mathbb{R}} \frac{d\xi_1 d\xi_2}{< \xi^3(z+F) >^{2b}} \leq C,$$

where C is a constant independent of ξ.

Proof: By Proposition 1 we can suppose $\xi_1 > 4$, so $(\xi_1 - 2) > \xi_1/2$. Using (10) and making change of variables as above, we have

$$J_1 \lesssim \frac{\xi^{2+4\rho}}{|\xi|^3} \int_4^\infty \frac{\xi_1^{4\rho}}{\xi_1\, l} d\xi_1 \leq C.$$

□

Proof of Lemma 4.
a) If $|\xi| \leq 1$.
Let $A_1 = \{(\xi_1, \xi_2)/|\xi_1| > 2, |\xi_2| > 2\}$, $A_2 = \{(\xi_1, \xi_2)/|\xi_1| \leq 2, |\xi_2| \leq 2\}$, $A_3 = \{(\xi_1, \xi_2)/|\xi_1| \leq 2, |\xi_2| > 2\}$ and $A_4 = \{(\xi_1, \xi_2)/|\xi_1| > 2, |\xi_2| \leq 2\}$ and consider $I(\xi, 0) = \sum_{j=1}^4 I_j(\xi, 0)$, where $I_j(\xi, 0)$ is defined in the region A_j. Obviously $I_2 \leq C$. In A_1 we have $|\xi - \xi_1| > |\xi_1|/2$ and $|\xi + \xi_2| > |\xi_2|/2$, therefore Lemma 3 gives $I_1 \leq C$. In A_3 we have $|\xi + \xi_2| > |\xi_2|/2$, and consequently

$$\begin{aligned} I_3(\xi,0) \lesssim & \frac{1}{< \xi >^{2\rho}} \int_{A_3} \frac{< \xi_2 >^{4\rho}\, d\xi_1 d\xi_2}{< (\xi_1 - \xi_2)\xi_2(\xi - \xi_1) >^{2b}} \\ = & \frac{1}{< \xi >^{2\rho}} \int_{A_3 \cap \{|\xi_1 - \xi_2| > |\xi_2|\}} + \frac{1}{< \xi >^{2\rho}} \int_{A_3 \cap \{|\xi_1 - \xi_2| \leq |\xi_2|\}} \\ = & I_{3,1}(\xi, 0) + I_{3,2}(\xi, 0). \end{aligned}$$

In the first integral, for $\rho < 1/4$, $b > 1/2$ we have

$$\begin{aligned} I_{3,1}(\xi,0) \lesssim & \frac{1}{< \xi >^{2\rho}} \int_{|\xi_2|>2} < \xi_2 >^{4\rho}\, d\xi_2 \int_{|\xi_1| \leq 2} \frac{d\xi_1}{< \xi_2^2(\xi - \xi_1) >^{2b}} \\ \lesssim & \frac{1}{< \xi >^{2\rho}} \int_{|\xi_2|>2} \frac{< \xi_2 >^{4\rho}\, d\xi_2}{\xi_2^2} \\ \leq & C. \end{aligned}$$

To estimate $I_{3,2}(\xi, 0)$ we make the change of variables $\eta_2 = \xi_1 - \xi_2$, $\eta_1 = \xi_1$ and as $|\xi_1| \leq 2$ we obtain the same estimate as that for $I_{3,1}(\xi, 0)$.

By symmetry we can estimate I_4 in the same manner as I_3.
b) If $|\xi| > 1$.
Let us consider $I(\xi, 0)$ in the form (11) and let $B_1 = \{|\xi_1 + \xi_2| > 4\}$ and $B_2 = \{|\xi_1 + \xi_2| \leq 4\}$, then $I(\xi, 0) = I_1(\xi) + I_2(\xi)$, where $I_j(\xi)$ is defined in B_j. In B_1 we have

$$|2 - (\xi_1 + \xi_2)| > |\xi_1 + \xi_2|/2, \ \ |1 - (\xi_1 + \xi_2)| \leq 5|\xi_1 + \xi_2|/4, \tag{15}$$

moreover $B_1 \subset \{|\xi_1| \geq 2\} \cup \{|\xi_2| \geq 2\} =: B_{1,1} \cup B_{1,2}$ and therefore $I_1(\xi, 0) \leq I_{1,1}(\xi) + I_{1,2}(\xi)$, where $I_{1,j}(\xi)$ is defined in $B_{1,j} \cap B_1$. In $B_{1,1}$ we have $|\xi_1|/2 \leq |1 - \xi_1| \leq 3|\xi_1|/2$, therefore using (15), we obtain that $I_{1,1}(\xi) \lesssim I(0, 0) \leq C$ if $\rho < 1/4$, $\rho + 1/3 < b$. In similar manner we have $I_{1,2}(\xi) \lesssim I(0, 0) \leq C$.

From definition of B_2 we have $H_\rho \lesssim < \xi >^{2\rho} < |\xi| + |\xi||\xi_2| >^{4\rho}$, so using symmetry and Propositions 1 and 2, we have $I_2(\xi) \leq C < \infty$ if $0 \leq \rho < 1/4$, $b > \rho + 1/3$.

Proof of Lemma 2.
Let $0 \le \rho < 1/4$, $7/12 < b < 11/12$. Using symmetry and Lemma 4 it is enough to prove

$$J = p(\xi, z) \int_0^\infty \int_{\mathbb{R}} \frac{H_\rho(\xi,\xi_1,\xi_2)d\xi_1 d\xi_2}{<\xi^3(z+F(\xi_1,\xi_2)>^{2b}} \le C < \infty.$$

By Lemma 4 we can suppose $|\xi|^3 z \ge 1$, because if $|\xi|^3 z < 1$ then

$$<\xi^3(z+F)>^{-2b} \le 2^{2b} <\xi^3 F>^{-2b}.$$

Also by symmetry we can suppose $|\xi_2| \le |\xi_1|$.
Therefore

$$H_\rho(\xi,\xi_1,\xi_2) \lesssim 1 + |\xi|^{6\rho} + |\xi|^{6\rho}|\xi_1|^{6\rho}. \tag{16}$$

Using Proposition 1 we can suppose $|\xi_1| > 4$ $(l^{-1} \le |\xi_1|^{-1})$.

a) If $|\xi||\xi_1| \le 1$.
We have $H_\rho \lesssim <\xi>^{6\rho}$ and therefore $J \le C < \infty$, by Proposition 1.
b) If $|\xi||\xi_1| > 1$.
i) If $|\xi_1|^3 \le z$, $|\xi_1| \le z^{1/3}$, we have $H_\rho(\xi,\xi_1,\xi_2) \lesssim 1 + |\xi|^{6\rho} + |z|^{2\rho/3}|\xi|^{6\rho}|\xi_1|^{4\rho}$. Therefore using (10), in this region we have

$$\begin{aligned}\frac{\xi^{2+6\rho}|z|^{2\rho/3}}{<\xi^3 z>^{2(1-b)}} \int_{1/|\xi|}^{|z|^{1/3}} |\xi_1|^{4\rho} d\xi_1 \int_{\mathbb{R}} \frac{d\eta}{<\xi^3\xi_1(\eta^2-l^2)>^{2b}} &\lesssim \frac{\xi^{2+6\rho}|z|^{2\rho/3}}{<\xi^3 z>^{2(1-b)}\ |\xi|^3} \int_{1/|\xi|}^\infty \frac{|\xi_1|^{4\rho} d\xi_1}{|\xi_1|^2} \\ &\lesssim \frac{(|\xi|^3 z)^{2\rho/3}}{<\xi^3 z>^{2(1-b)}} \\ &\le C.\end{aligned}$$

ii) If $|\xi_1|^3 \ge z$, $|\xi_1| \ge z^{1/3}$, we can proceed as follows.
By Lemma 4 we can suppose $|z+F| \le |F|/2$, so $|F| \le 2z$, $|(2-(\xi_1+\xi_2))\xi_1\xi_2| \le 2z$. This implies that $|1-\xi_2||1-(\xi_1+\xi_2)| \lesssim 1 + |\xi_1| + z^{2/3}$.
Therefore

$$\begin{aligned}H_\rho \lesssim (<\xi>^{4\rho} + |\xi|^{6\rho}) + |\xi|^{4\rho}|\xi_1|^{4\rho} + |\xi|^{6\rho}|\xi_1|^{2\rho} + |\xi|^{4\rho}|\xi_1|^{2\rho} + |\xi|^{6\rho}|\xi_1|^{4\rho} \\ + |\xi|^{4\rho} z^{4\rho/3} + |\xi|^{6\rho} z^{4\rho/3} + |\xi|^{6\rho} z^{4\rho/3}|\xi_1|^{2\rho} = \sum_{j=1}^{8} l_j.\end{aligned}$$

We have,

$$\frac{|\xi|^{6\rho}}{<\xi>^{2\rho}} \le |\xi|^{4\rho}. \tag{17}$$

To estimate the term that contains $l_1 = <\xi>^{4\rho} + |\xi|^{6\rho}$, we use (17) and Proposition 1.

For terms l_j, $j = 2, \ldots, 5$, we use (17) and Propositions 1 and 2 if $|\xi| > 1$. If $|\xi| < 1$, we integrate in the region $\xi_1 > 1/|\xi|$ as above.

In $l_6 = |\xi|^{4\rho} z^{4\rho/3}$, we have

$$\frac{|\xi|^2 |\xi|^{4\rho} z^{4\rho/3}}{< \xi^3 z >^{2(1-b)} |\xi|^3 < \xi >^{2\rho}} \int_{z^{1/3}}^{\infty} \frac{d\xi_1}{\xi_1^2} \lesssim \frac{1}{(|\xi|^3 z)^{(1-4\rho)/3}} \leq C.$$

We estimate $l_7 = |\xi|^{6\rho} z^{4\rho/3}$, as in l_6 using (17).
Finally in $l_8 = |\xi|^{6\rho} z^{4\rho/3} |\xi_1|^{2\rho}$, we have

$$\frac{|\xi|^{2+6\rho} z^{4\rho/3}}{< \xi >^{2\rho} |\xi|^3} \int_{z^{1/3}}^{\infty} \frac{|\xi_1|^{2\rho} d\xi_1}{\xi_1^2} \lesssim \frac{(|\xi|^3 z)^{(6\rho-1)/3}}{< \xi^3 z >^{2(1-b)}} \leq C.$$

□

4. Proof of Theorem 2.

Consider a cut-off function $\psi \in \mathcal{C}^\infty$, such that $0 \leq \psi \leq 1$,

$$\psi(t) = \begin{cases} 1 & \text{if } |t| \leq 1 \\ 0 & \text{if } |t| \geq 2, \end{cases} \tag{18}$$

and let $\psi_T(t) := \psi(t/T)$. To prove Theorem 2 we need the following result.

Proposition 3. *Let $-1/2 < b' \leq 0 \leq b \leq b' + 1$, $T \in [0,1]$, then*

$$\|\psi_1(t) U(t) u_0\|_{\{s,b\}} = C \|u_0\|_{\mathbf{H^s}} \tag{19}$$

$$\|\psi_T(t) \int_0^t U(t-t') F(t',\cdot)) dt'\|_{\{s,b\}} \leq C T^{1-b+b'} \|F(u)\|_{\{s,b'\}}, \tag{20}$$

where $F(u) := i\gamma |u|^2 u$.

Proof: The proof of (19) is obvious. The proof of (20) is practically done in [6]. □

Let us consider (5) in its equivalent integral form

$$u(t) = U(t) u_0 - \int_0^t U(t-t') F(u)(t',\cdot)) dt'. \tag{21}$$

Note that, if for all $t \in \mathbb{R}$, $u(t)$ satisfies:

$$u(t) = \psi_1(t) U(t) u_0 - \psi_T(t) \int_0^t U(t-t') F(u)(t',\cdot)) dt', \tag{22}$$

then $u(t)$ satisfies (21) in $[-T, T]$. Let $a > 0$ and

$$X_a = \{v \in X^{s,b}; \|v\|_{s,b} \leq a\}. \tag{23}$$

For $v \in X_a$ fixed, let us define

$$\Phi(v) = \psi_1(t) U(t) u_0 - \psi_T(t) \int_0^t U(t-t') F(v)(t',\cdot)) dt'.$$

Let $\epsilon = 1 - b + b' > 0$, using Proposition 3 and Theorem 3 we obtain

$$\begin{aligned}\|\Phi(v)\|_{s,b} \leq & C\|u_0\|_{\mathbf{H^s}} + CT^{\epsilon}\|F(v)\|_{s,b'} \\ \leq & C\|u_0\|_{\mathbf{H^s}} + CT^{\epsilon}M^3 \\ \leq & M,\end{aligned}$$

where we took $M = 2C\|u_0\|_{\mathbf{H^s}}$, $T^{\epsilon} \leq 1/(2CM^2)$.

We can prove that Φ is a contraction in an analogous manner. The proof of the Theorem 1 follows by using a standard argument, see for example [9, 10].

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IMECC-UNICAMP, Caixa Postal:6065, 13083-859, Barão Geraldo, Campinas, SP, Brazil
E-mail address: carvajal@ime.unicamp.br