SMOLUCHOWSKI'S EQUATION: RATE OF CONVERGENCE OF THE MARCUS-LUSHNIKOV PROCESS

by

Eduardo Cepeda and Nicolas Fournier

Abstract. — We derive a satisfying rate of convergence of the Marcus-Lushnikov process toward the solution to Smoluchowski's coagulation equation. Our result applies to a class of homogeneous-like coagulation kernels with homogeneity degree ranging in $(-\infty, 1]$. It relies on the use of a Wasserstein-type distance, which has shown to be particularly well-adapted to coalescence phenomena. It was introduced in [**7**].

Mathematics Subject Classification (2000): 60H30, 45K05.

Keywords: Smoluchowski's coagulation equation, Marcus-Lushnikov process, Interacting stochastic particle systems.

Contents

1. Introduction

We are interested in coalescence which is a widespread phenomenon: it arises in physics, chemistry, astrophysics, biology and mathematics.

We consider a possibly infinite system of particles, each particle being fully identified by its mass ranging in the set of positive real numbers. The only mechanism taken into account is the coalescence of two particles with masses x and y into a single one with mass $x + y$ at some given rate (the "coagulation kernel") $K(x, y) = K(y, x) \geq 0$.

– We can consider a system of microscopic particles and the following system of differential equations for the concentrations $\mu_t(x)$ of particles of mass $x = 1, 2, 3, ...$ at time $t \in [0, +\infty)$:

$$\partial_t \mu_t(x) = \frac{1}{2} \sum_{y=1}^{x-1} K(y, x-y)\mu_t(y)\mu_t(x-y) - \mu_t(x) \sum_{y=1}^{+\infty} K(x, y)\mu_t(y). \tag{1.1}$$

The first sum in (1.1) on the right corresponds to coagulation of smaller particles to produce one of mass x, whereas the second sum corresponds to removal of particles of mass x as they in turn coagulate to produce larger particles.

Analogous integro-differential equations allow us to consider a continuum of masses x. In this case the system can also be described by the concentration $\mu_t(x)$ of particles of mass $x \in (0, +\infty)$ at time $t \in [0, +\infty)$. Then $\mu_t(x)$ solves a nonlinear equation:

$$\partial_t \mu_t(x) = \frac{1}{2} \int_0^x K(y, x-y)\mu_t(y)\mu_t(x-y)dy - \mu_t(x) \int_0^{+\infty} K(x, y)\mu_t(y)dy. \tag{1.2}$$

Equation (1.2) is known as the continuous Smoluchowski coagulation equation and (1.1) is its discrete version.

– When the particles are macroscopic and when the rate of coagulation is not infinitesimal, the frame of study of the dynamics of such a system is stochastic. When the initial state consists of a finite number of macroscopic particles, the stochastic coalescent obviously exists (see [**1**]) and it is known as the Marcus-Lushnikov process.

In preceding works several results have been obtained on the existence and uniqueness of weak solutions to Smoluchowski's coagulation equation. The general framework was formulated in [**15**] who obtained some remarkable well-posedness results. In [**7**], homogeneous-like kernels are considered and it has been seen that the well-posedness holds in the class of measures having a finite moment of order the degree of homogeneity of the coagulation kernel.

Aldous [**1**] presents the Marcus-Lushnikov process as an approximation for the solution of Smoluchowski's equation (see [**14, 13**] for further information). Since then some results on convergence have been obtained in [**15**] and in [**10**], see also [**6**]. A class of stochastic algorithms in which the number of particles remains constant in time was introduced in [**3**] and has been extended to the discrete coagulation-fragmentation case in [**11**].

We investigate the rate of convergence of the Marcus-Lushnikov process to the solution of the Smoluchowski coagulation equation as the number of particles tends to infinity. This problem is interesting because on the one hand it has a physical meaning: the Smoluchowski equation is often derived by passing to the limit in the Marcus-Lushnikov process, and on the other hand from a numerical point of view: this stochastic process can be simulated exactly. Thus it seems natural to use it in order to approximate the solution to Smoluchowski's coagulation equation.

Our study is based on the use of a specific Wasserstein-type distance d_λ between the solution to Smoluchowski's equation and its stochastic approximation. This distance depends on the homogeneity parameter λ of the coagulation kernel. This specific distance has been introduced in [**7**] to prove some results on the well-posedness of the Smoluchowski coagulation equation and in [**5, 8**] to study the stochastic coalescent. The result of the present work applies to a family of homogeneous-like coagulation kernels. These kernels are of particular importance in applications see Table 1 in [**1**] or the list provided in [**7**].

We point out that since we are using a finite particle system to approximate the evolution in time of the solution to the Smoluchowski equation which describes an infinite particle system, it is necessary to dispose of a mechanism to construct an initial condition for the Marcus-Lushnikov process from a general measure-valued initial condition of Smoluchowski's equation. This initial condition needs to satisfy, on the one hand, a convergence condition to assure the convergence of the stochastic process to the solution to Smoluchowski's equation for all time t as the number of particles grows (the usual condition of weak convergence is replaced by convergence in the sense of the distance we use), and on the other hand it must obey a rate of convergence in order to control the overall rate of convergence of such an approximation.

Very roughly, we consider a homogeneous-like coagulation kernel with degree of homogeneity $\lambda \in (-\infty, 1] \setminus \{0\}$ (including $K(x, y) = (x + y)^\lambda$). For $(\mu_t)_{t\geq 0}$ the solution to the corresponding Smoluchowski's equation and for $(\mu_t^n)_{t\geq 0}$ the corresponding Marcus-Lushnikov process, we prove that

$$\sup_{t\in[0,T]} \mathbb{E}\left[d_\lambda(\mu_t^n, \mu_t)\right] \leq \frac{C_T}{\sqrt{n}},$$

as soon as μ_0 satisfies some technical conditions and for a good choice of the initial state of the Marcus-Lushnikov process μ_0^n of the form $\frac{1}{n}\sum_{k=1}^N \delta_{x_k}$. We can make the following remarks.

1. Recalling the *Central Limit Theorem* (CLT), this rate of convergence seems to be optimal, since the convergence of μ_t^n to μ_t is a generalized Law of Large Numbers.

2. In [**7**] it has been seen that only one moment is demanded to show the well-posedness for the Smoluchowski equation. In the present work, we need to demand more moments, but we believe that it is very difficult to avoid such conditions.

3. The only works giving an explicit result on the rate of convergence of the Marcus-Lushnikov process toward the solution to Smoluchowski's coagulation equation, known by us, are:

- Norris [**15**], who gives an estimate using a "Large Deviations" approach for the discrete case ($\mathrm{supp}(\mu_0) \subset \mathbb{N}^*$).
- Deaconu, Fournier and Tanré [**2**], where a CLT-type result is shown for the discrete case and for a bounded coagulation kernel K, furthermore in this work a different particle system is used.
- Kolokoltsov [**12**], who gives a CLT result for the discrete case with a coagulation kernel satisfying $K(x, y) \leq c(1 + \sqrt{x})(1 + \sqrt{y})$ and for the continuous case when K is two times differentiable with all its derivatives bounded. Unfortunately the case $K(x, y) = (x + y)^\lambda$ is excluded for any value of $\lambda \in (-\infty, 1] \setminus \{0\}$.

Our work thus gives the first result on the rate of convergence covering the continuous case for some homogeneous kernels.

For the case $\lambda < 0$ we follow the ideas found in [**7**], but for the case $\lambda \in (0, 1]$ the proof is much more difficult and the calculations are faced in a completely different way. Namely we use the Itô formula for an approximation of the absolute value function and handle very delicately the resulting terms.

The paper is organized as follows: in Section 2 we give the notation and definitions we use in this document, in Section 3 we state our main result. The proof is developed in Sections 4, 5 and 6. We give also a method to construct an initial condition for the Marcus-Lushnikov process in Section 7 and we conclude the document giving some technical details which are useful all along the paper in Appendix A.

2. Notation, Assumptions and Definitions

In this section we present our assumptions, give the definition of weak solutions to Smoluchowski's coagulation equation and then we recall the dynamics of the Marcus-Lushnikov process.

Notation 2.1. — *We denote by $\mathcal{M}^+$ the space of non-negative Radon measures on $(0,+\infty)$. For a measure μ and a function ϕ, we set $\langle \mu(dx)\,,\,\phi(x)\rangle = \int_0^{+\infty}\phi(x)\mu(dx)$. We also define the operator A for all measurable functions $\phi:(0,+\infty)\to\mathbb{R}$, by*

$$(A\phi)(x,y) = \phi(x+y) - \phi(x) - \phi(y) \quad \forall\,(x,y)\in(0,+\infty)^2. \tag{2.1}$$

Finally, we will use the notation $x\wedge y = \min\{x,y\}$ and $x\vee y = \max\{x,y\}$ for $(x,y)\in(0,+\infty)^2$.

We consider a coagulation kernel $K:(0,+\infty)\times(0,+\infty)\to[0,+\infty)$, symmetric i.e. $K(x,y)=K(y,x)$ for $(x,y)\in(0,+\infty)^2$. We further assume it belongs to $W^{1,\infty}\left((\varepsilon,1/\varepsilon)^2\right)$ for every $\varepsilon\in(0,1)$ and one of the following conditions $\forall\,(x,y)\in(0,+\infty)^2$:

$$\lambda\in(-\infty,0),\quad K(x,y)\le\kappa_0\,(x+y)^\lambda \text{ and } \left(x^\lambda+y^\lambda\right)|\partial_x K(x,y)|\le\kappa_1 x^{\lambda-1}y^\lambda, \tag{2.2}$$

$$\lambda\in(0,1],\quad K(x,y)\le\kappa_0\,(x+y)^\lambda \text{ and } \left(x^\lambda\wedge y^\lambda\right)|\partial_x K(x,y)|\le\kappa_1 x^{\lambda-1}y^\lambda, \tag{2.3}$$

$$\lambda\in(0,1],\quad K(x,y)\le\kappa_0\,(x\wedge y)^\lambda \text{ and } \left(x^\lambda\wedge y^\lambda\right)|\partial_x K(x,y)|\le\kappa_1 x^{\lambda-1}y^\lambda, \tag{2.4}$$

for some positive constants κ_0 and κ_1. We refer to [**7**] for a list of physical kernels satisfying conditions (2.2) and (2.3). Remark that for any $\lambda\in(-\infty,1]\setminus\{0\}$, $K(x,y)=(x+y)^\lambda$ satisfies (2.2) or (2.3).

Definition 2.2. — *Consider $\lambda\in(-\infty,1]\setminus\{0\}$. For $\mu\in\mathcal{M}^+$, we set:*

$$M_\lambda(\mu) = \int_0^{+\infty} x^\lambda\mu(dx) \qquad \textit{and} \qquad \mathcal{M}^+_\lambda = \{\nu\in\mathcal{M}^+ : M_\lambda(\nu)<+\infty\}. \tag{2.5}$$

For $\mu\in\mathcal{M}^+$, we set, for $x\in(0,+\infty)$:

$$F^\mu(x) = \int_0^{+\infty}\mathbb{1}_{(x,+\infty)}(y)\,\mu(dy) \quad \textit{and} \quad G^\mu(x) = \int_0^{+\infty}\mathbb{1}_{(0,x]}(y)\,\mu(dy). \tag{2.6}$$

We define the distance on $\mathcal{M}^+_\lambda$ as

$$d_\lambda(\mu,\tilde\mu) = \int_0^{+\infty} x^{\lambda-1}|E(x)|dx, \tag{2.7}$$

where $E(x)=G^\mu(x)-G^{\tilde\mu}(x)$ if $\lambda\in(-\infty,0)$ and $E(x)=F^\mu(x)-F^{\tilde\mu}(x)$ if $\lambda\in(0,1]$.

We remark that d_λ is well-defined on $\mathcal{M}^+_\lambda$. Indeed we have $d_\lambda(\mu,\tilde\mu)\le\frac{1}{|\lambda|}M_\lambda(\mu+\tilde\mu)$ for $\lambda\in(-\infty,1]\setminus\{0\}$. See [**4**] for a deeper study of this distance in the discrete and continuous cases.

We excluded the case $\lambda=0$ for two reasons. First, d_0 is not well-defined on $\mathcal{M}^+_0$. Next, when trying to extend our study to this case, we are not able to obtain a better result than those of Kolokoltsov [**12**].

Definition 2.3. — *For $\lambda \in (-\infty, 1] \setminus \{0\}$ we introduce the spaces of test functions needed to define weak solutions:*

$$\text{if } \lambda \in (-\infty,0): \quad \mathcal{H}_\lambda = \left\{\phi : (0,+\infty) \to \mathbb{R} \text{ such that } \sup_{x>0} x^{-\lambda}|\phi(x)| < +\infty\right\},$$

$$\text{if } \lambda \in (0,1]: \quad \mathcal{H}_\lambda = \left\{\phi : (0,+\infty) \to \mathbb{R} \text{ such that } \sup_{x>0}(1+x)^{-\lambda}|\phi(x)| < +\infty\right\},$$

$$\text{if } \lambda \in (0,1]: \quad \mathcal{H}^e_\lambda = \left\{\phi : (0,+\infty) \to \mathbb{R} \text{ such that } \sup_{x>0} x^{-\lambda}|\phi(x)| < +\infty\right\}.$$

It is necessary to introduce the space $\mathcal{H}^e_\lambda$ to study the case (2.4).

2.1. The Smoluchowski coagulation equation. — The weak formulation of the Smoluchowski coagulation equation is given by

$$\frac{d}{dt}\langle \mu_t(dx), \phi(x)\rangle = \frac{1}{2}\langle \mu_t(dx)\mu_t(dy), (A\phi)(x,y)K(x,y)\rangle, \tag{2.8}$$

see Notation 2.1. This is a general formulation and it embraces the two previous equations : if μ_0 is discrete (i.e. $\mathrm{supp}(\mu_0) \subset \mathbb{N}^*$), then this corresponds to the "discrete coagulation equation" (1.1), while when μ_0 is continuous (i.e. $\mu_0(dx) = \mu_0(x)dx$), this corresponds to the "continuous coagulation equation" (1.2). Formulation (2.8) is standard, see [**15**].

Definition 2.4. — *Let $\lambda \in (-\infty, 1] \setminus \{0\}$, a coagulation kernel K satisfying either (2.2), (2.3) or (2.4), and $\mu^{in} \in \mathcal{M}^+_\lambda$. We will then say that $(\mu_t)_{t\geq 0} \subset \mathcal{M}^+$ is a (μ^{in}, K, λ)-weak solution to Smoluchowski's equation if the following conditions are verified:*

(i) $\mu_0 = \mu^{in}$,

(ii) the application $t \longmapsto \langle \mu_t(dx), \phi(x)\rangle$ is differentiable on $[0, +\infty)$ and satisfies (2.8) for each $\phi \in \mathcal{H}_\lambda$ (cases (2.2) and (2.3)) or for each $\phi \in \mathcal{H}^e_\lambda$ (case (2.4)),

(iii) for all $T \in [0, +\infty)$

$$\sup_{s\in[0,T]} M_\alpha(\mu_s) < +\infty, \tag{2.9}$$

for $\alpha = \lambda$ (cases (2.2) and (2.4)) or for $\alpha = 0,\ 2\lambda$ (case (2.3)).

We demand more finite moments of μ_0 than in [**7**] to assure the convergence of the Marcus-Lushnikov process. According to the hypothesis on the kernel (2.2), (2.3) or (2.4) together with (2.9) and Lemma A.1, the integrals in the weak formulation (2.8) are absolutely convergent and bounded with respect to $t \in [0, T]$ for every T.

Under (2.2) or (2.4), the existence and uniqueness of such weak solutions have been established in [**7**] for any $\mu^{in} \in \mathcal{M}^+_\lambda$. Under (2.3), the existence and uniqueness of weak solutions satisfying (2.9) with $\alpha = \lambda$ have also been checked in [**7**] for any $\mu^{in} \in \mathcal{M}^+_\lambda$. Using furthermore Proposition A.4, we immediately deduce the existence and uniqueness of weak solutions under (2.3), in the sense of Definition 2.4, for any $\mu^{in} \in \mathcal{M}^+_0 \cap \mathcal{M}^+_{2\lambda}$.

2.2. The Marcus-Lushnikov process. — The Marcus-Lushnikov process describes the stochastic Markov evolution of a finite particle system of coalescing particles. We consider a coagulation kernel K and a finite particle system initially consisting of $N \geq 2$ particles of masses $x_1, \cdots, x_N \in (0, +\infty)$. We assume that the system evolves according to the following dynamics: each pair of particles (of masses x and y) coalesce (i.e. disappears and forms a new particle of mass $x + y$) with a rate proportional to $K(x, y)$.

Let $n \in \mathbb{N}$ and we assign to all particles the weight $1/n$. We define now rigorously the Marcus-Lushnikov process to be used.

Definition 2.5. — *We consider a coagulation kernel K, $n \in \mathbb{N}$ and an initial state $\mu_0^n = \frac{1}{n}\sum_{i=1}^N \delta_{x_i}$, with $x_1, \cdots, x_N \in (0,+\infty)$.*

The Marcus-Lushnikov process $(\mu_t^n)_{t\geq 0}$ associated with (n, K, μ_0^n) is a Markov $\mathcal{M}^+$-valued càdlàg process satisfying:

(i) $(\mu_t^n)_{t\geq 0}$ takes its values in $\left\{\frac{1}{n}\sum_{i=1}^k \delta_{y_i}; k \leq N,\, y_i > 0\right\}$.

(ii) Its infinitesimal generator is given, for all mesurable functions $\Psi : \mathcal{M}^+ \to \mathbb{R}$ and all states $\mu = \frac{1}{n}\sum_{i=1}^k \delta_{y_i}$ by

$$L\Psi(\mu) = \sum_{1\leq i<j\leq k} \left\{\Psi\left[\mu + n^{-1}\left(\delta_{y_i+y_j} - \delta_{y_i} - \delta_{y_j}\right)\right] - \Psi[\mu]\right\} \frac{K(y_i,y_j)}{n}.$$

This process is known to be well-defined and unique, see [**1, 15**]. We will use the following classical representation of the Marcus-Lushnikov process (see e.g. [**5, 8**]): there is a Poisson measure $J(dt, d(i,j), dz)$ on $[0,+\infty) \times \{(i,j) \in \mathbb{N}^2, i < j\} \times [0,+\infty)$ with intensity measure $dt\left[\sum_{k<l}\delta_{(k,l)}\left(d(i,j)\right)\right]dz$, such that for any measurable function $\phi : (0,+\infty) \to \mathbb{R}$

$$\begin{aligned} \langle \mu_t^n(dx), \phi(x)\rangle \;&=\; \langle \mu_0^n(dx), \phi(x)\rangle + \int_0^t \int_{i<j} \int_0^{+\infty} \frac{1}{n}\left[\phi\left(X_{s-}^i + X_{s-}^j\right) - \phi\left(X_{s-}^i\right) - \phi\left(X_{s-}^j\right)\right] \\ &\qquad \mathbb{1}_{\left\{z \leq \frac{K\left(X_{s-}^i, X_{s-}^j\right)}{n}\right\}} \mathbb{1}_{\{j\leq N(s-)\}} J(ds, d(i,j), dz), \end{aligned} \tag{2.10}$$

where $\mu_t^n = \frac{1}{n}\sum_{k=1}^{N(t)} \delta_{X_t^k}$, $N(t)$ being the (non-increasing) number of particles at time t.

This can be written using the compensated Poisson measure related to J:

$$\begin{aligned} \langle \mu_t^n(dx), \phi(x)\rangle \;&=\; \langle \mu_0^n(dx), \phi(x)\rangle + \frac{1}{2}\int_0^t \langle \mu_s^n(dx)\mu_s^n(dy), (A\phi)(x,y)K(x,y)\rangle ds \\ &\quad - \frac{1}{2n}\int_0^t \langle \mu_s^n(dx), (A\phi)(x,x)K(x,x)\rangle ds \\ &\quad + \int_0^t \int_{i<j}\int_0^{+\infty} \frac{1}{n}\,(A\phi)\left(X_{s-}^i, X_{s-}^j\right) \mathbb{1}_{\left\{z \leq \frac{K\left(X_{s-}^i, X_{s-}^j\right)}{n}\right\}} \mathbb{1}_{\{j\leq N(s-)\}} \\ &\qquad \tilde{J}(ds, d(i,j), dz), \end{aligned} \tag{2.11}$$

where the operator A is defined in (2.1). The third term on the right-hand side is issued from the impossibility of coalescence of a particle with itself.

3. Results

We state in this section our main result. We also state as a proposition the construction of a sequence of initial conditions for the Marcus-Lushnikov processes and finally comment on our results.

Theorem 3.1. — *We consider* $\lambda \in (-\infty, 1] \setminus \{0\}$ *and a coagulation kernel* K *satisfying either (2.2), (2.3) or (2.4). Let* $\mu_0 \in \mathcal{M}^+$ *and* $(\mu_t)_{t\geq 0}$ *the* (μ_0, K, λ)*-weak solution to Smoluchowski's equation. Let* μ_0^n *be deterministic and of the form* $\frac{1}{n}\sum_{i=1}^{N} \delta_{x_i}$ *and denote by* $(\mu_t^n)_{t\geq 0}$ *the associated* (n, K, μ_0^n)*-Marcus-Lushnikov process. Let* $\varepsilon > 0$.

- *Assume (2.2) or (2.4) and that* μ_0 *belongs to* $\mathcal{M}_\lambda^+ \cap \mathcal{M}_{2\lambda+\tilde{\varepsilon}}^+$, *where* $\tilde{\varepsilon} = sgn(\lambda) \times \varepsilon$. *Then for any* $T > 0$,

$$\mathbb{E}\left[\sup_{t\in[0,T]} d_\lambda(\mu_t^n, \mu_t)\right] \leq \left[d_\lambda(\mu_0^n, \mu_0) + \frac{(1+T)C_{\lambda,\varepsilon}}{\sqrt{n}}\Big(M_\lambda(\mu_0^n) + M_{2\lambda+\tilde{\varepsilon}}(\mu_0^n)\Big)\right] \times \exp\left[TC_{\lambda,\varepsilon} M_\lambda(\mu_0^n + \mu_0)\right],$$

where $C_{\lambda,\varepsilon}$ *is a positive constant depending only on* λ, ε *and* κ_0, *and* κ_1.

- *Assume (2.3) and that* $\mu_0 \in \mathcal{M}_0^+ \cap \mathcal{M}_{\gamma+\varepsilon}^+$ *where* $\gamma = \max\{2\lambda, 4\lambda - 1\}$. *Then for any* $T > 0$,

$$\sup_{t\in[0,T]} \mathbb{E}\left[d_\lambda(\mu_t^n, \mu_t)\right] \leq \left[d_\lambda(\mu_0^n, \mu_0) + \frac{(1+T)C_{\lambda,\varepsilon}}{\sqrt{n}}\Big(1 + \left[M_0(\mu_0^n + \mu_0)\right]^2 + \left[M_{\gamma+\varepsilon}(\mu_0^n + \mu_0)\right]^2\Big)\right] \times \exp\left[TC_{\lambda,\varepsilon} M_\lambda(\mu_0^n + \mu_0)\right],$$

where $C_{\lambda,\varepsilon}$ *is a positive constant depending only on* λ, ε, κ_0 *and* κ_1.

Now we present the proposition giving a d_λ-approximation of the initial condition.

Proposition 3.2. — *Let* $\lambda \in (-\infty, 1] \setminus \{0\}$, $n \in \mathbb{N}$ *and* μ_0 *a non negative Radon measure on* $(0, +\infty)$ *such that* $\mu_0 \in \mathcal{M}_\lambda^+ \cap \mathcal{M}_{2\lambda}^+$. *The measure* μ_0 *is supposed to be either atomless or discrete (*$\mathrm{supp}(\mu_0) \subset \mathbb{N}^*$*). Then, there exists a positive measure* μ_0^n *of the form* $\frac{1}{n}\sum_{i=1}^{N_n} \delta_{x_i}$ *such that:*

$$d_\lambda(\mu_0^n, \mu_0) \leq \frac{C_\lambda}{\sqrt{n}},$$

where the constant C_λ *depends only on* λ, $M_\lambda(\mu_0)$ *and* $M_{2\lambda}(\mu_0)$. *We also have*

$$M_\alpha(\mu_0^n) \leq M_\alpha(\mu_0),$$

for all $\alpha \leq 0$ *if* $\lambda \in (-\infty, 0)$ *and for all* $\alpha \geq 0$ *if* $\lambda \in (0, 1]$. *Furthermore, if* $M_0(\mu_0) < +\infty$, *then*

$$N_n \leq n\, M_0(\mu_0).$$

The estimate of the parameter N_n (initial number of particles) may be useful to study the numerical cost of the simulation.

Gathering Theorem 3.1 and Proposition 3.2, we deduce the following statement.

Corollary 3.3. — *We consider* $\lambda \in (-\infty, 1] \setminus \{0\}$, $\varepsilon > 0$ *and a coagulation kernel* K *satisfying either (2.2), (2.3) or (2.4). Let* $\mu_0 \in \mathcal{M}^+$ *be either atomless or discrete (supp(*μ_0*)* $\subset \mathbb{N}^*$*), and* $(\mu_t)_{t\in[0,+\infty)}$ *the* (μ_0, K, λ)*-weak solution to Smoluchowski's equation. Then it is possible to build a family of initial conditions* $\mu_0^n = \frac{1}{n}\sum_{k=1}^{N_n} \delta_{x_i}$ *such that, for* $(\mu_t^n)_{t\geq 0}$ *the corresponding* (n, K, μ_0^n)*-Marcus-Lushnikov process,*

- *under (2.2) or (2.4), if* μ_0 *belongs to* $\mathcal{M}_\lambda^+ \cap \mathcal{M}_{2\lambda+\tilde{\varepsilon}}^+$, *where* $\tilde{\varepsilon} = sgn(\lambda) \times \varepsilon$, *then for any* $T > 0$,

$$\mathbb{E}\left[\sup_{t\in[0,T]} d_\lambda(\mu_t^n, \mu_t)\right] \leq \frac{C_T}{\sqrt{n}},$$

where C_T is a positive constant depending only on T, λ, ε, κ_0, κ_1 and μ_0;

- *under (2.3), if $\mu_0 \in \mathcal{M}_0^+ \cap \mathcal{M}_{\gamma+\varepsilon}^+$ where $\gamma = \max\{2\lambda, 4\lambda - 1\}$, then for any $T > 0$,*

$$\sup_{t\in[0,T]} \mathbb{E}\left[d_\lambda(\mu_t^n, \mu_t)\right] \quad \le \quad \frac{C_T}{\sqrt{n}},$$

where C_T is a positive constant depending only on T, λ, ε, κ_0, κ_1 and μ_0.

This last statement is quite satisfying since it provides a rate of convergence in $\frac{1}{\sqrt{n}}$ and it applies to a large class of homogeneous kernels presenting singularities for small or large masses. We probably require more finite moments than really needed but this does not seem to be a real problem for applications.

We have followed the ideas found in [**7**] to prove the case (2.2) and the special case (2.4) of Theorem 3.1. The case (2.3) is much more subtle and difficult. For this case we have applied the Itô formula and manipulated very carefully each term. By the moment it is not possible to put the "sup" into the expectation since it is very important to use the sign of the terms and to take advantage of some cancelations.

Proposition 3.2 presents the proof of the existence of a d_λ-approximation of a general non-negative measure μ_0 (we consider measures μ_0 wich are interesting for the Smoluchowski's equation) by a discrete measure μ_0^n (a finite sum of Dirac's deltas) as a construction procedure. This construction is very useful from a numerical point of view since it gives a measure that will be set as the initial state for the Marcus-Lushnikov process.

4. Negative Case

In the whole section, we assume that K satisfies (2.2) for some fixed $\lambda \in (-\infty, 0)$. We fix $\varepsilon > 0$, and we assume that $\mu_0 \in \mathcal{M}_\lambda^+ \cap \mathcal{M}_{2\lambda-\varepsilon}^+$. We denote by $(\mu_t)_{t\ge 0}$ the unique (μ_0, K, λ)-weak solution to the Smoluchowski equation. We also consider the (n, K, μ_0^n)-Marcus Lushnikov process, for some given initial condition $\mu_0^n = \frac{1}{n}\sum_{i=1}^N \delta_{x_i}$.

We introduce, for $t \ge 0$, the quantity $E_n(t,x) = G^{\mu_t^n}(x) - G^{\mu_t}(x)$ as defined in (2.6). We take the test function $\phi(v) = \mathbb{1}_{(0,x]}(v)$. Since $\sup_{v>0} v^{-\lambda}|\phi(v)| = x^{-\lambda} < +\infty$, we deduce that $\phi \in \mathcal{H}_\lambda$. Computing the difference between equations (2.11) and (2.8), we get

$$\begin{aligned} E_n(t,x) \quad = \quad & E_n(0,x) + \frac{1}{2}\int_0^t \left\langle \mu_s^n(dv)\mu_s^n(dy) - \mu_s(dv)\mu_s(dy), \left(A\mathbb{1}_{(0,x]}\right)(v,y)K(v,y)\right\rangle ds \\ & - \frac{1}{2n}\int_0^t \langle \mu_s^n(dv), \left(A\mathbb{1}_{(0,x]}\right)(v,v)K(v,v)\rangle ds \\ & + \int_0^t \int_{i<j}\int_0^{+\infty} \frac{1}{n}\left(A\mathbb{1}_{(0,x]}\right)\left(X_{s-}^i, X_{s-}^j\right)\mathbb{1}_{\left\{z\le \frac{K\left(X_{s-}^i, X_{s-}^j\right)}{n}\right\}}\mathbb{1}_{\{j\le N(s-)\}} \\ & \qquad \tilde{J}(ds, d(i,j), dz). \end{aligned} \tag{4.1}$$

We take the absolute value and integrate against $x^{\lambda-1}dx$ on $(0, +\infty)$:

$$d_\lambda(\mu_t^n, \mu_t) \quad \le \quad d_\lambda(\mu_0^n, \mu_0) + A_1(t) + A_2(t) + A_3(t), \tag{4.2}$$

where

$$\begin{aligned} A_1(t) &= \frac{1}{2}\int_0^{+\infty} x^{\lambda-1}\left|\int_0^t \langle \mu_s^n(dv)\mu_s^n(dy) - \mu_s(dv)\mu_s(dy), \left(A\mathbb{1}_{(0,x]}\right)(v,y)K(v,y)\rangle ds\right| dx, \\ A_2(t) &= \frac{1}{2n}\int_0^{+\infty} x^{\lambda-1}\left|\int_0^t \langle \mu_s^n(dv), \left(A\mathbb{1}_{(0,x]}\right)(v,v)K(v,v)\rangle\, ds\right| dx, \\ A_3(t) &= \int_0^{+\infty} x^{\lambda-1}\left|\frac{1}{n}\int_0^t\int_{i<j}\int_0^{+\infty} \left(A\mathbb{1}_{(0,x]}\right)\left(X_{s-}^i, X_{s-}^j\right)\mathbb{1}_{\left\{z\le \frac{K\left(X_{s-}^i, X_{s-}^j\right)}{n}\right\}}\right. \\ &\qquad\qquad \left.\mathbb{1}_{\{j\le N(s-)\}}\tilde{J}(ds, d(i,j), dz)\right| dx. \end{aligned}$$

Now we are going to search for a good upper bound for each term.

Term $A_1(t)$.

The term $A_1(t)$, according to the symmetry of the kernel, can be written as:

$$A_1(t) = \frac{1}{2}\int_0^{+\infty} x^{\lambda-1}\left|\int_0^t\int_0^{+\infty}\int_0^{+\infty} K(v,y)\left[\mathbb{1}_{(0,x]}(v+y) - \mathbb{1}_{(0,x]}(v) - \mathbb{1}_{(0,x]}(y)\right]\right. \left.\left(\mu_s^n - \mu_s\right)(dv)\left(\mu_s^n + \mu_s\right)(dy)ds\right| dx. \tag{4.3}$$

We use the Fubini theorem and Lemma A.2:

$$\int_0^t\int_0^{+\infty}\int_0^{+\infty} K(v,y)\left[\mathbb{1}_{(0,x]}(v+y) - \mathbb{1}_{(0,x]}(v) - \mathbb{1}_{(0,x]}(y)\right]\left(\mu_s^n - \mu_s\right)(dv)\left(\mu_s^n + \mu_s\right)(dy)ds$$

$$\begin{aligned} &= \int_0^t\int_0^{+\infty}\int_0^{+\infty}\Big\{K(x-y,y)\mathbb{1}_{(0,x]}(v+y) - K(x,y)\mathbb{1}_{(0,x]}(v) \\ &\qquad - \int_v^{+\infty}\partial_x K(z,y)\left[\mathbb{1}_{(0,x]}(z+y) - \mathbb{1}_{(0,x]}(z) - \mathbb{1}_{(0,x]}(y)\right]dz\Big\}\left(\mu_s^n - \mu_s\right)(dv) \\ &\qquad\qquad \left(\mu_s^n + \mu_s\right)(dy)ds \end{aligned}$$

$$\begin{aligned} &= \int_0^t\int_0^{+\infty} K(x-y,y)\left[\mathbb{1}_{x>y}\int_0^{+\infty}\mathbb{1}_{(0,x-y]}(v)\left(\mu_s^n - \mu_s\right)(dv)\right]\left(\mu_s^n + \mu_s\right)(dy)ds \\ &\quad - \int_0^t\int_0^{+\infty} K(x,y)\left[\int_0^{+\infty}\mathbb{1}_{(0,x]}(v)\left(\mu_s^n - \mu_s\right)(dv)\right]\left(\mu_s^n + \mu_s\right)(dy)ds \\ &\quad - \int_0^t\int_0^{+\infty}\int_0^{+\infty}\partial_x K(z,y)\left[\mathbb{1}_{(0,x]}(z+y) - \mathbb{1}_{(0,x]}(z) - \mathbb{1}_{(0,x]}(y)\right] \\ &\qquad\qquad \left[\int_0^{+\infty}\mathbb{1}_{(0,z]}(v)\left(\mu_s^n - \mu_s\right)(dv)\right]dz\left(\mu_s^n + \mu_s\right)(dy)ds \end{aligned}$$

$$
\begin{aligned}
&= \int_0^t \int_0^{+\infty} K(x-y,y)\,[\mathbb{1}_{x>y}\, E_n(s,x-y)]\,(\mu_s^n+\mu_s)\,(dy)ds \\
&\quad - \int_0^t \int_0^{+\infty} K(x,y)\,[E_n(s,x)]\,(\mu_s^n+\mu_s)\,(dy)ds \\
&\quad - \int_0^t \int_0^{+\infty} \int_0^{+\infty} \partial_x K(z,y)\,\big[\mathbb{1}_{(0,x]}(z+y)-\mathbb{1}_{(0,x]}(z)-\mathbb{1}_{(0,x]}(y)\big] \\
&\qquad\qquad [E_n(s,z)]\,dz\,(\mu_s^n+\mu_s)\,(dy)ds.
\end{aligned}
$$

According to the bound

$$
\big|\mathbb{1}_{(0,x]}(z+y)-\mathbb{1}_{(0,x]}(z)-\mathbb{1}_{(0,x]}(y)\big| \le 2\,\mathbb{1}_{(0,x]}(z\wedge y), \tag{4.4}
$$

and using (2.2), we deduce:

$$
\begin{aligned}
A_1(t) \;\le\;& \frac{\kappa_0}{2}\int_0^t \int_0^{+\infty} \int_y^{+\infty} x^{\lambda-1} x^{\lambda} |E_n(s,x-y)|\,dx\,(\mu_s^n+\mu_s)\,(dy)\,ds \\
&+\frac{\kappa_0}{2}\int_0^t \int_0^{+\infty} \int_0^{+\infty} x^{\lambda-1}(x+y)^{\lambda} |E_n(s,x)|\,dx\,(\mu_s^n+\mu_s)\,(dy)\,ds \\
&+\int_0^t \int_0^{+\infty} \int_0^{+\infty} |\partial_x K(z,y)||E_n(s,z)| \left[\int_0^{+\infty} x^{\lambda-1}\mathbb{1}_{(0,x]}(z\wedge y)dx\right] dz\,(\mu_s^n+\mu_s)\,(dy)\,ds.
\end{aligned}
$$

For the first integral we use the change of variable $x \mapsto w+y$ and $(w+y)^{\lambda-1}\,(w+y)^{\lambda} \le w^{\lambda-1}y^{\lambda}$. For the second integral $(x+y)^{\lambda} \le y^{\lambda}$. Finally for the third integral, we observe that $\int_0^{+\infty} x^{\lambda-1}\mathbb{1}_{(0,x]}(z\wedge y)dx = \frac{(z\wedge y)^{\lambda}}{|\lambda|} \le \frac{z^{\lambda}+y^{\lambda}}{|\lambda|}$. Using (2.2) again, this implies

$$
\begin{aligned}
A_1(t) \;\le\;& \frac{\kappa_0}{2}\int_0^t ds \int_0^{+\infty} w^{\lambda-1}|E_n(s,w)|\,dw \int_0^{+\infty} y^{\lambda}\,(\mu_s^n+\mu_s)\,(dy) \\
&+\frac{\kappa_0}{2}\int_0^t ds \int_0^{+\infty} x^{\lambda-1}|E_n(s,x)|\,dx \int_0^{+\infty} y^{\lambda}\,(\mu_s^n+\mu_s)\,(dy) \\
&+\frac{\kappa_1}{|\lambda|}\int_0^t ds \int_0^{+\infty} z^{\lambda-1}|E_n(s,z)|\,dz \int_0^{+\infty} y^{\lambda}\,(\mu_s^n+\mu_s)\,(dy).
\end{aligned}
$$

The resulting bound for $A_1(t)$ is:

$$
A_1(t) \le \left(\kappa_0 + \frac{\kappa_1}{|\lambda|}\right) \int_0^t d_\lambda(\mu_s^n,\mu_s)\, M_\lambda(\mu_s^n+\mu_s)\,ds. \tag{4.5}
$$

Term $A_2(t)$.

We use $\left|\left(A\mathbb{1}_{(0,x]}\right)(v,v)\right| = \left|\mathbb{1}_{(0,x]}(2v) - 2\,\mathbb{1}_{(0,x]}(v)\right| = \mathbb{1}_{\{0<v\le \frac{x}{2}\}} + 2\,\mathbb{1}_{\{\frac{x}{2}<v\le x\}} \le 2\,\mathbb{1}_{\{v\le x\}}$. This gives

$$
\begin{aligned}
A_2(t) \;&\le\; \frac{1}{n}\int_0^{+\infty} x^{\lambda-1}\int_0^t \int_0^{+\infty} K(v,v)\mathbb{1}_{\{v\le x\}}\mu_s^n(dv)\,ds\,dx \\
&\le\; \frac{1}{n}\int_0^{+\infty} \int_0^t \kappa_0(2v)^{\lambda}\frac{v^{\lambda}}{|\lambda|}\mu_s^n(dv)\,ds \\
&\le\; \frac{2^{\lambda}\kappa_0}{n\,|\lambda|}\int_0^t M_{2\lambda}(\mu_s^n)\,ds. \qquad (4.6)
\end{aligned}
$$

We used (2.2).

Term $A_3(t)$.

We will bound the expectation of this term using its bracket, for this we consider:

$$\begin{aligned}
&\mathbb{E}\left[\left(\frac{1}{n}\int_0^t\int_{i<j}\int_0^{+\infty}(A\mathbb{1}_{(0,x]})\left(X^i_{s-},X^j_{s-}\right)\mathbb{1}_{\left\{z\leq\frac{K\left(X^i_{s-},X^j_{s-}\right)}{n}\right\}}\mathbb{1}_{\{j\leq N(s-)\}}\tilde{J}(ds,d(i,j),dz)\right)^2\right]\\
&\quad=\mathbb{E}\left[\int_0^t\frac{1}{n^2}\sum_{i<j\leq N(s)}\frac{K\left(X^i_s,X^j_s\right)}{n}\left[\mathbb{1}_{(0,x]}\left(X^i_s+X^j_s\right)-\mathbb{1}_{(0,x]}\left(X^i_s\right)-\mathbb{1}_{(0,x]}\left(X^j_s\right)\right]^2ds\right]\\
&\quad\leq\frac{4}{n}\mathbb{E}\left[\int_0^t\sum_{i<j\leq N(s)}\frac{K\left(X^i_s,X^j_s\right)}{n^2}\mathbb{1}_{(0,x]}\left(X^i_s\wedge X^j_s\right)ds\right]\\
&\quad\leq\frac{2}{n}\mathbb{E}\left[\int_0^t\left\langle\mu^n_s(dv)\mu^n_s(dy)\,,\,K(v,y)\,\left[\mathbb{1}_{(0,x]}(v)+\mathbb{1}_{(0,x]}(y)\right]\right\rangle ds\right]\\
&\quad\leq\frac{4\,\kappa_0}{n}\mathbb{E}\left[\int_0^t\left\langle\mu^n_s(dv)\mu^n_s(dy)\,,\,(v+y)^\lambda\,\mathbb{1}_{(0,x]}(v)\right\rangle ds\right].
\end{aligned}$$

We have used (4.4), (2.2) and a symmetry argument. We consider now the submartingale (absolute value of a martingale):

$$S_t(x)=\left|\frac{1}{n}\int_0^t\int_{i<j}\int_0^{+\infty}\left(A\mathbb{1}_{(0,x]}\right)\left(X^i_{s-},X^j_{s-}\right)\mathbb{1}_{\left\{z\leq\frac{K\left(X^i_{s-},X^j_{s-}\right)}{n}\right\}}\mathbb{1}_{\{j\leq N(s-)\}}\tilde{J}(ds,d(i,j),dz)\right|.$$

According to the Cauchy-Schwartz and Doob inequalities we have:

$$\mathbb{E}\left[\sup_{r\in[0,t]}S_r(x)\right]\leq\left(\mathbb{E}\left[\sup_{r\in[0,t]}\left(S_r(x)\right)^2\right]\right)^{\frac{1}{2}}\leq 2\left(\mathbb{E}\left[\left(S_t(x)\right)^2\right]\right)^{\frac{1}{2}}.$$

Therefore, we obtain the following bound for the expectation of $A_3(t)$:

$$\begin{aligned}
\mathbb{E}\left[\sup_{s\in[0,t]}A_3(s)\right]\quad\leq\quad&\frac{4\sqrt{\kappa_0}}{\sqrt{n}}\int_0^{+\infty}x^{\lambda-1}\\
&\left\{\mathbb{E}\left[\int_0^t\left\langle\mu^n_s(dv)\mu^n_s(dy)\,,\,(v+y)^\lambda\,\mathbb{1}_{(0,x]}(v)\right\rangle ds\right]\right\}^{\frac{1}{2}}dx.
\end{aligned}\tag{4.7}$$

Following the value of x we use different bounds:

On the one hand, for $x \leq 1$ we have $\mathbb{1}_{(0,x]}(v) \leq \left(\frac{v}{x}\right)^{2\lambda-\varepsilon}$ and using the bound $(v+y)^\lambda v^{2\lambda-\varepsilon} \leq v^{2\lambda-\varepsilon} y^\lambda$, we obtain:

$$
\begin{aligned}
\int_0^1 x^{\lambda-1} \left\{ \mathbb{E}\left[\int_0^t \left\langle \mu_s^n(dv)\mu_s^n(dy)\,,\,(v+y)^\lambda \,\mathbb{1}_{(0,x]}(v)\right\rangle ds \right] \right\}^{\frac{1}{2}} dx \\
&\leq \int_0^1 x^{\lambda-1} \left\{ \mathbb{E}\left[\int_0^t \left\langle \mu_s^n(dv)\mu_s^n(dy)\,,\,\frac{v^{2\lambda-\varepsilon}y^\lambda}{x^{2\lambda-\varepsilon}}\right\rangle ds \right] \right\}^{\frac{1}{2}} dx \\
&= \int_0^1 x^{\frac{\varepsilon}{2}-1} dx \;\left\{ \mathbb{E}\left[\int_0^t \left\langle \mu_s^n(dv)\mu_s^n(dy)\,,\,v^{2\lambda-\varepsilon}y^\lambda\right\rangle ds \right] \right\}^{\frac{1}{2}} \\
&= \frac{2}{\varepsilon} \left\{ \mathbb{E}\left[\int_0^t M_\lambda(\mu_s^n)\, M_{2\lambda-\varepsilon}(\mu_s^n)\, ds \right] \right\}^{\frac{1}{2}}. \qquad (4.8)
\end{aligned}
$$

On the other hand, for $x > 1$ we have $\mathbb{1}_{(0,x]}(v) \leq \left(\frac{v}{x}\right)^{\lambda}$ and using the bound $(v+y)^\lambda v^\lambda \leq v^\lambda y^\lambda$, we obtain:

$$
\begin{aligned}
\int_1^{+\infty} x^{\lambda-1} \left\{ \mathbb{E}\left[\int_0^t \left\langle \mu_s^n(dv)\mu_s^n(dy)\,,\,(v+y)^\lambda \,\mathbb{1}_{(0,x]}(v)\right\rangle ds \right] \right\}^{\frac{1}{2}} dx \\
&\leq \int_1^{+\infty} x^{\lambda-1} \left\{ \mathbb{E}\left[\int_0^t \left\langle \mu_s^n(dv)\mu_s^n(dy)\,,\,\frac{v^{\lambda}y^\lambda}{x^{\lambda}}\right\rangle ds \right] \right\}^{\frac{1}{2}} dx \\
&= \int_1^{+\infty} x^{\frac{\lambda}{2}-1} dx \;\left\{ \mathbb{E}\left[\int_0^t \left\langle \mu_s^n(dv)\mu_s^n(dy)\,,\,v^{\lambda}y^\lambda\right\rangle ds \right] \right\}^{\frac{1}{2}} \\
&= \frac{2}{|\lambda|} \left\{ \mathbb{E}\left[\int_0^t \left[M_\lambda(\mu_s^n)\right]^2\, ds \right] \right\}^{\frac{1}{2}}. \qquad (4.9)
\end{aligned}
$$

Then, writing the right-hand side integral of (4.7) as the sum of the integrals on $x \in (0,1]$ and $x \in (1,+\infty)$, gathering (4.8) and (4.9), we get

$$
\begin{aligned}
\mathbb{E}\left[\sup_{s\in[0,t]} A_3(s)\right] \quad \leq \quad & \frac{8\sqrt{\kappa_0}}{\sqrt{n}} \Bigg\{ \frac{1}{\varepsilon} \left(\mathbb{E}\left[\int_0^t M_\lambda(\mu_s^n) M_{2\lambda-\varepsilon}(\mu_s^n)\, ds \right] \right)^{\frac{1}{2}} \\
& + \frac{1}{|\lambda|} \left(\mathbb{E}\left[\int_0^t \left[M_\lambda(\mu_s^n)\right]^2\, ds \right] \right)^{\frac{1}{2}} \Bigg\}. \qquad (4.10)
\end{aligned}
$$

Conclusion.

Gathering (4.2), (4.5), (4.6) and (4.10), we have:

$$\begin{aligned}\mathbb{E}\left[\sup_{s\in[0,t]} d_\lambda(\mu_s^n,\mu_s)\right] &\leq \mathbb{E}\left[d_\lambda(\mu_0^n,\mu_0)+\sup_{s\in[0,t]}A_1(s)+\sup_{s\in[0,t]}A_2(s)+\sup_{s\in[0,t]}A_3(s)\right]\\ &\leq d_\lambda(\mu_0^n,\mu_0)+\left(\kappa_0+\frac{\kappa_1}{|\lambda|}\right)\int_0^t \mathbb{E}\left[d_\lambda(\mu_s^n,\mu_s)\,M_\lambda(\mu_s^n+\mu_s)\right]ds\\ &\quad+\frac{2^\lambda\kappa_0}{n\,|\lambda|}\int_0^t\mathbb{E}\left[M_{2\lambda}(\mu_s^n)\right]\,ds\\ &\quad+\frac{8\sqrt{\kappa_0}}{\sqrt{n}}\left\{\frac{1}{\varepsilon}\left(\mathbb{E}\left[\int_0^t M_\lambda(\mu_s^n)\,M_{2\lambda-\varepsilon}(\mu_s^n)\,ds\right]\right)^{\frac12}\right.\\ &\qquad\left.+\frac{1}{|\lambda|}\left(\mathbb{E}\left[\int_0^t\left[M_\lambda(\mu_s^n)\right]^2\,ds\right]\right)^{\frac12}\right\}.\end{aligned}$$

According to Proposition A.4 –*(a)*, $M_\alpha(\mu_t^n+\mu_t)\leq M_\alpha(\mu_0^n+\mu_0)$ a.s. for any $\alpha\in(-\infty,0)$. Since μ_0^n is deterministic, we get:

$$\begin{aligned}\mathbb{E}\left[\sup_{s\in[0,t]} d_\lambda(\mu_t^n,\mu_t)\right] &\leq d_\lambda(\mu_0^n,\mu_0)+\left(\kappa_0+\frac{\kappa_1}{|\lambda|}\right)M_\lambda(\mu_0^n+\mu_0)\int_0^t\mathbb{E}\left[d_\lambda(\mu_s^n,\mu_s)\right]ds\\ &\quad+\frac{2^\lambda\kappa_0}{n\,|\lambda|}M_{2\lambda}(\mu_0^n)\,t+\frac{8\sqrt{\kappa_0}}{\sqrt{n}}\left[\frac{1}{\varepsilon}\big(M_\lambda(\mu_0^n)\,M_{2\lambda-\varepsilon}(\mu_0^n)\big)^{\frac12}+\frac{1}{|\lambda|}M_\lambda(\mu_0^n)\right]t^{\frac12}.\end{aligned}\tag{4.11}$$

Finally, since $\sqrt{ab}\leq a+b$ and since $M_{2\lambda}(\mu_0^n)\leq M_\lambda(\mu_0^n)+M_{2\lambda-\varepsilon}(\mu_0^n)$, we use the Gronwall lemma to obtain

$$\begin{aligned}\mathbb{E}\left[\sup_{t\in[0,T]} d_\lambda(\mu_t^n,\mu_t)\right] &\leq \left[d_\lambda(\mu_0^n,\mu_0)+\frac{C_1}{\sqrt{n}}M_\lambda(\mu_0^n)+\frac{C_2}{\sqrt{n}}M_{2\lambda-\varepsilon}(\mu_0^n))\right]\\ &\quad\times\exp\left[T\left(\kappa_0+\frac{\kappa_1}{|\lambda|}\right)M_\lambda(\mu_0^n+\mu_0)\right],\end{aligned}\tag{4.12}$$

where $C_1=\frac{2^\lambda T\kappa_0}{|\lambda|}+\frac{8(\varepsilon+|\lambda|)}{\varepsilon\,|\lambda|}\sqrt{T\kappa_0}$ and $C_2=\frac{2^\lambda T\kappa_0}{|\lambda|}+\frac{8}{\varepsilon}\sqrt{T\kappa_0}$.

This concludes the proof of Theorem 3.1 under (2.2).

5. Positive Case

In the whole section, we assume that K satisfies (2.4) for some fixed $\lambda\in(0,1]$. We fix $\varepsilon>0$, and we assume that $\mu_0\in\mathcal{M}_0^+\cap\mathcal{M}_{\gamma+\varepsilon}^+$ where $\gamma=\max\{2\lambda,\,4\lambda-1\}$. We denote by $(\mu_t)_{t\geq0}$ the unique (μ_0,K,λ)-weak solution to the Smoluchowski equation. We also consider the (n,K,μ_0^n)-Marcus Lushnikov process, for some given initial condition $\mu_0^n=\frac1n\sum_{i=1}^N\delta_{x_i}$.

We assume without loss of generality, for $\lambda\in(0,1/2)$, that $\varepsilon<\frac12-\lambda$. Indeed, if $\varepsilon\geq\frac12-\lambda$, it suffices to consider $\tilde\varepsilon<\frac12-\lambda$, to apply Theorem 3.1 with $\tilde\varepsilon$, and to use the bound $M_{2\lambda+\tilde\varepsilon}(\mu_0^n+\mu_0)\leq M_0(\mu_0^n+\mu_0)+M_{2\lambda+\varepsilon}(\mu_0^n+\mu_0)$ to conclude.

We first present a lemma of which the proof is developed in the appendix.

Lemma 5.1. — *We introduce, for $x \in (0,+\infty)$, the following application:*

$$\theta^n_{(x)} = \frac{1}{\sqrt{n}} \mathbb{1}_{(0,1]}(x) + \frac{x^{-2\lambda-\varepsilon}}{\sqrt{n}} \mathbb{1}_{(1,+\infty)}(x). \tag{5.1}$$

Then,

(i) $\int_0^{+\infty} x^{\lambda-1}\theta^n_{(x)} dx \le \frac{2}{\lambda\sqrt{n}}$,

(ii) $\int_0^{+\infty} x^{2\lambda-1}\theta^n_{(x)} dx \le \frac{\lambda+\varepsilon}{\lambda\varepsilon\sqrt{n}}$,

(iii) for $(v,y) \in (0,+\infty)^2$

$$v^\lambda \int_0^{+\infty} \frac{x^{\lambda-1}}{\theta^n_{(x)}} \left(\mathbb{1}_{x<v\wedge y} + \mathbb{1}_{v\vee y<x<v+y}\right) dx \le \frac{2\sqrt{n}}{\lambda} v^\lambda y^\lambda$$
$$+\sqrt{n}\left(2^{2\lambda+\varepsilon} + \frac{1}{\lambda}\right)\left[(v\wedge y)^{2\lambda}(v\vee y)^{2\lambda+\varepsilon}\mathbb{1}_{\lambda\in(0,1/2)} + (v\wedge y)(v\vee y)^{4\lambda+\varepsilon-1}\mathbb{1}_{\lambda\in[1/2,1]}\right].$$

We set $E_n(t,x) = F^{\mu^n_t}(x) - F^{\mu_t}(x)$ as defined in (2.6), for $x \in (0,+\infty)$. We take the test function $\phi(v) = \mathbb{1}_{(x,+\infty)}(v)$. Since $\sup_{v>0}\frac{|\phi(v)|}{(1+v)^\lambda} = (1+x)^{-\lambda} < +\infty$, we deduce that $\phi \in \mathcal{H}_\lambda$. Again, computing the difference between equations (2.11) and (2.8) and using a symmetry argument for the first integral, we get

$$\begin{aligned} E_n(t,x) &= E_n(0,x) + \frac{1}{2}\int_0^t \langle(\mu^n_s - \mu_s)(dv)(\mu^n_s + \mu_s)(dy), \left(A\mathbb{1}_{(x,+\infty)}\right)(v,y)K(v,y)\rangle ds \\ &\quad - \frac{1}{2n}\int_0^t \langle \mu^n_s(dv), \left(A\mathbb{1}_{(x,+\infty)}\right)(v,v)K(v,v)\rangle ds \\ &\quad + \int_0^t \int_{i<j}\int_0^{+\infty} \frac{1}{n}\left(A\mathbb{1}_{(x,+\infty)}\right)\left(X^i_{s-}, X^j_{s-}\right)\mathbb{1}_{\left\{z \le \frac{K\left(X^i_{s-}, X^j_{s-}\right)}{n}\right\}}\mathbb{1}_{\{j\le N(s-)\}} \\ &\qquad \tilde{J}(ds, d(i,j), dz). \end{aligned} \tag{5.2}$$

According to Lemma A.2, we can write the first integral as:

$$\begin{aligned} &\int_0^t\int_0^{+\infty}\int_0^{+\infty} K(v,y)\left(A\mathbb{1}_{(x,+\infty)}\right)(v,y)(\mu^n_s-\mu_s)(dv)(\mu^n_s+\mu_s)(dy)ds \\ &\quad = \int_0^t\int_0^{+\infty}\int_0^{+\infty}\Big\{\mathbb{1}_{x>y}K(x-y,y)\mathbb{1}_{(x,+\infty)}(v+y) - K(x,y)\mathbb{1}_{(x,+\infty)}(v) \\ &\qquad + \int_0^v \partial_x K(z,y)\left(A\mathbb{1}_{(x,+\infty)}\right)(z,y)\,dz\Big\}(\mu^n_s-\mu_s)(dv)(\mu^n_s+\mu_s)(dy)ds \\ &\quad = \int_0^t\int_0^{+\infty} K(x-y,y)\left[\mathbb{1}_{x>y}\int_0^{+\infty}\mathbb{1}_{(x-y,+\infty)}(v)(\mu^n_s-\mu_s)(dv)\right](\mu^n_s+\mu_s)(dy)ds \\ &\qquad - \int_0^t\int_0^{+\infty} K(x,y)\left[\int_0^{+\infty}\mathbb{1}_{(x,+\infty)}(v)(\mu^n_s-\mu_s)(dv)\right](\mu^n_s+\mu_s)(dy)ds \\ &\qquad + \int_0^t\int_0^{+\infty}\int_0^{+\infty}\partial_x K(z,y)\left(A\mathbb{1}_{(x,+\infty)}\right)(z,y) \\ &\qquad\qquad \left[\int_0^{+\infty}\mathbb{1}_{(z,+\infty)}(v)(\mu^n_s-\mu_s)(dv)\right]dz\,(\mu^n_s+\mu_s)(dy)ds. \end{aligned}$$

Recalling that $E_n(s,x)=\int_0^{+\infty}\mathbb{1}_{(x,+\infty)}(v)\left(\mu_s^n-\mu_s\right)(dv)$, we deduce that,

$$\begin{aligned}
E_n(t,x) &= E_n(0,x)+\frac{1}{2}\int_0^t\left[\overline{B}_1(s,x)+\overline{B}_2(s,x)+\overline{B}_3(s,x)\right]ds \\
&\quad+\int_0^t\int_{i<j}\int_0^{+\infty}\frac{1}{n}\left(A\mathbb{1}_{(x,+\infty)}\right)\left(X^i_{s-},X^j_{s-}\right)\mathbb{1}_{\left\{z\le\frac{K\left(X^i_{s-},X^j_{s-}\right)}{n}\right\}}\mathbb{1}_{\{j\le N(s-)\}} \\
&\qquad\tilde{J}(ds,d(i,j),dz),
\end{aligned}\tag{5.3}$$

where:

$$\begin{aligned}
\overline{B}_1(s,x) &= \int_0^{+\infty}\left[\mathbb{1}_{x>y}K(x-y,y)E_n(s,x-y)-E_n(s,x)K(x,y)\right]\left(\mu_s^n+\mu_s\right)(dy),\\
\overline{B}_2(s,x) &= \int_0^{+\infty}\int_0^{+\infty}\partial_xK(z,y)\left(A\mathbb{1}_{(x,+\infty)}\right)(z,y)E_n(s,z)dz\left(\mu_s^n+\mu_s\right)(dy),\\
\overline{B}_3(s,x) &= -\frac{1}{n}\int_0^{+\infty}K(v,v)\left[\mathbb{1}_{(x,+\infty)}(2v)-2\,\mathbb{1}_{(x,+\infty)}(v)\right]\mu_s^n(dv).
\end{aligned}$$

Now, we apply the Itô formula to $\varphi_\theta(E_n(t,x))$, where $\varphi_\theta(\cdot)\in\mathcal{C}^2(\mathbb{R})$ is an approximation of the absolute value function $|\cdot|$. This function is chosen in such a way that:

$$\left\{\begin{array}{ll}
\varphi_\theta(u)=|u|\ \text{ if } |u|>\theta; & |u|\le\varphi_\theta(u)\le|u|+\theta\ \ \forall u\in\mathbb{R};\\
|\varphi'_\theta(u)|\le 1\ \ \forall u\in\mathbb{R}; & sgn(u\varphi'_\theta(u))=1\ \forall u\in\mathbb{R}_*;\\
|\varphi''_\theta(u)|\le\dfrac{2}{\theta}\mathbb{1}_{\{|u|<\theta\}}\ \ \forall u\in\mathbb{R}. &
\end{array}\right.\tag{5.4}$$

Furthermore, we consider for θ the function defined by (5.1). We fix $x\in(0,+\infty)$ and apply the Itô formula to $\varphi_{\theta^n_{(x)}}(E_n(t,x))$ (see for exemple [**9**]),

$$\begin{aligned}
\varphi_{\theta^n_{(x)}}(E_n(t,x)) &= \varphi_{\theta^n_{(x)}}(E_n(0,x))\\
&\quad+\frac{1}{2}\int_0^t\left[\overline{B}_1(s,x)+\overline{B}_2(s,x)+\overline{B}_3(s,x)\right]\varphi'_{\theta^n_{(x)}}(E_n(s,x))ds\\
&\quad+M(t,x)+\overline{B}_4(t,x),
\end{aligned}\tag{5.5}$$

where

$$\begin{aligned}
M(t,x) &= \int_0^t\int_{i<j}\int_0^{+\infty}\frac{1}{n}\left(A\mathbb{1}_{(x,+\infty)}\right)\left(X^i_{s-},X^j_{s-}\right)\mathbb{1}_{\left\{z\le\frac{K\left(X^i_{s-},X^j_{s-}\right)}{n}\right\}}\mathbb{1}_{\{j\le N(s-)\}}\\
&\qquad\varphi'_{\theta^n_{(x)}}(E_n(s-,x))\tilde{J}(ds,d(i,j),dz),\\
\overline{B}_4(t,x) &= \int_0^t\int_{i<j}\int_0^{+\infty}\left\{\varphi_{\theta^n_{(x)}}\left(E_n(s-,x)+\frac{1}{n}\left(A\mathbb{1}_{(x,+\infty)}\right)\left(X^i_{s-},X^j_{s-}\right)\right)-\varphi_{\theta^n_{(x)}}(E_n(s-,x))\right.\\
&\quad\left.-\frac{1}{n}\left(A\mathbb{1}_{(x,+\infty)}\right)\left(X^i_{s-},X^j_{s-}\right)\varphi'_{\theta^n_{(x)}}(E_n(s-,x))\right\}\mathbb{1}_{\left\{z\le\frac{K\left(X^i_{s-},X^j_{s-}\right)}{n}\right\}}\mathbb{1}_{\{j\le N(s-)\}}\\
&\qquad J(ds,d(i,j),dz).
\end{aligned}$$

Observe that, for all $x\ge 0$, $M(t,x)$ is a martingale whose expectation is equal to zero.

Now, we study the θ^n-approximation of $d_\lambda(\mu_t^n, \mu_t)$: $\int_0^{+\infty} x^{\lambda-1}\varphi_{\theta^n_{(x)}}(E_n(t,x))dx$. According to (5.4) and Lemma 5.1 –*(i)*, we have:

$$d_\lambda(\mu_s^n, \mu_s) \le \int_0^{+\infty} x^{\lambda-1}\varphi_{\theta^n_{(x)}}(E_n(s,x))dx \le d_\lambda(\mu_s^n, \mu_s) + \frac{2}{\lambda\sqrt{n}}. \tag{5.6}$$

Consider (5.5), integrate each term against $x^{\lambda-1}dx$ on $(0+\infty)$, take the expectation:

$$\begin{aligned} \mathbb{E}\left[d_\lambda(\mu_t^n, \mu_t)\right] &\le \int_0^{+\infty} x^{\lambda-1}\mathbb{E}\left[\varphi_{\theta^n_{(x)}}(E_n(t,x))\right]dx \\ &= \int_0^{+\infty} x^{\lambda-1}\varphi_{\theta^n_{(x)}}(E_n(0,x))dx + \mathbb{E}\left[B_1(t)+B_2(t)+B_3(t)+B_4(t)\right], \end{aligned} \tag{5.7}$$

where

$$\begin{aligned} B_1(t) &= \frac{1}{2}\int_0^{+\infty}\int_0^t x^{\lambda-1}\overline{B}_1(s,x)\varphi'_{\theta^n_{(x)}}(E_n(s,x))\,ds\,dx, \\ B_2(t) &= \frac{1}{2}\int_0^{+\infty}\int_0^t x^{\lambda-1}\overline{B}_2(s,x)\varphi'_{\theta^n_{(x)}}(E_n(s,x))\,ds\,dx, \\ B_3(t) &= \frac{1}{2}\int_0^{+\infty}\int_0^t x^{\lambda-1}\overline{B}_3(s,x)\varphi'_{\theta^n_{(x)}}(E_n(s,x))\,ds\,dx, \\ B_4(t) &= \int_0^{+\infty} x^{\lambda-1}\overline{B}_4(t,x)\,dx. \end{aligned}$$

We now study each term separately.

Term $B_1(t)$.

We use the Fubini theorem to obtain:

$$\begin{aligned} B_1(t) = \frac{1}{2}\int_0^t\int_0^{+\infty}\Bigg[&\int_0^{+\infty} \mathbb{1}_{x>y}\, x^{\lambda-1}\,\varphi'_{\theta^n_{(x)}}(E_n(s,x))\,E_n(s,x-y)K(x-y,y)dx \\ &- \int_0^{+\infty} x^{\lambda-1}\,\varphi'_{\theta^n_{(x)}}(E_n(s,x))\,E_n(s,x)K(x,y)\,dx\Bigg]\left(\mu_s^n+\mu_s\right)(dy)\,ds. \end{aligned}$$

Recalling (5.4), we immediately deduce that $\varphi'_{\theta^n_{(x)}}(E_n(s,x))\,E_n(s,x-y) \le |E_n(s,x-y)|$, and $\varphi'_{\theta^n_{(x)}}(E_n(s,x))\,E_n(s,x) = \left|\varphi'_{\theta^n_{(x)}}(E_n(s,x))\right|\,|E_n(s,x)|$. Therefore, using the change of variable $x \mapsto u+y$ in the first integral, we get:

$$\begin{aligned} B_1(t) &\le \frac{1}{2}\int_0^t\int_0^{+\infty}\Bigg[\int_0^{+\infty}(u+y)^{\lambda-1}\,|E_n(s,u)|\,K(u,y)\,du \\ &\qquad - \int_0^{+\infty} x^{\lambda-1}\left|\varphi'_{\theta^n_{(x)}}(E_n(s,x))\right|\,|E_n(s,x)|\,K(x,y)\,dx\Bigg]\left(\mu_s^n+\mu_s\right)(dy)\,ds \\ &= \frac{1}{2}\int_0^t\int_0^{+\infty}\int_0^{+\infty} K(z,y)\,|E_n(s,z)|\left[(z+y)^{\lambda-1} - \left|\varphi'_{\theta^n_{(z)}}(E_n(s,z))\right|\,z^{\lambda-1}\right]dz \\ &\qquad\qquad \left(\mu_s^n+\mu_s\right)(dy)\,ds. \end{aligned}$$

Recall again (5.4). Since $|E_n(s,z)| \geq \theta^n_{(z)}$ implies $\left|\varphi'_{\theta^n_{(z)}}\left(E_n(s,z)\right)\right| = 1$, and since $(z+y)^{\lambda-1} - z^{\lambda-1} \leq 0$,

$$\begin{aligned} |E_n(s,z)| \left[(z+y)^{\lambda-1} - |\varphi'_{\theta^n_{(z)}}\left(E_n(s,z)\right)|\, z^{\lambda-1}\right] &\leq |E_n(s,z)|\,(z+y)^{\lambda-1}\mathbb{1}_{\left\{|E_n(s,z)|<\theta^n_{(z)}\right\}} \\ &\leq \theta^n_{(z)}(z+y)^{\lambda-1}. \end{aligned}$$

Therefore, using (2.3):

$$\begin{aligned} B_1(t) &\leq \frac{\kappa_0}{2}\int_0^t\int_0^{+\infty}\int_0^{+\infty}\theta^n_{(z)}\,(z+y)^{2\lambda-1}\,dz\,(\mu^n_s+\mu_s)\,(dy)\,ds \\ &\leq \frac{\kappa_0}{2}\int_0^t\int_0^{+\infty}\left[\int_0^{+\infty}\theta^n_{(z)}z^{2\lambda-1}dz + y^\lambda\int_0^{+\infty}\theta^n_{(z)}z^{\lambda-1}dz\right](\mu^n_s+\mu_s)\,(dy)\,ds. \end{aligned}$$

We used $(z+y)^{2\lambda-1} = (z+y)^\lambda(z+y)^{\lambda-1} \leq (z^\lambda+y^\lambda)\,z^{\lambda-1}$. Finally, according to Lemma 5.1–*(i)* and *(ii)*, we get:

$$\begin{aligned} B_1(t) &\leq \frac{\kappa_0}{2}\int_0^t\int_0^{+\infty}\left[\frac{2(\lambda+\varepsilon)}{\lambda\varepsilon\sqrt{n}}\left(1+y^\lambda\right)\right](\mu^n_s+\mu_s)\,(dy)\,ds \\ &\leq \frac{\kappa_0(\lambda+\varepsilon)}{\lambda\varepsilon\sqrt{n}}\int_0^t\left[M_0(\mu^n_s+\mu_s)+M_\lambda(\mu^n_s+\mu_s)\right]ds. \end{aligned} \tag{5.8}$$

Term $B_2(t)$.

First, observe that

$$\begin{aligned} \left|\left(A\mathbb{1}_{(x,+\infty)}\right)(z,y)\right| &= \left|\mathbb{1}_{(x,+\infty)}(z+y) - \mathbb{1}_{(x,+\infty)}(z) - \mathbb{1}_{(x,+\infty)}(y)\right| \\ &= \mathbb{1}_{\{x\in(0,z\wedge y)\}} + \mathbb{1}_{\{x\in(z\vee y,z+y)\}}\,, \end{aligned} \tag{5.9}$$

whence,

$$\begin{aligned} \int_0^{+\infty}x^{\lambda-1}\left|\left(A\mathbb{1}_{(x,+\infty)}\right)(z,y)\right|dx &= \int_0^{z\wedge y}x^{\lambda-1}dx + \int_{z\vee y}^{z+y}x^{\lambda-1}dx \\ &\leq \frac{2}{\lambda}(z\wedge y)^\lambda. \end{aligned} \tag{5.10}$$

Thus, recalling (5.4), we get:

$$\begin{aligned} B_2(t) &\leq \frac{1}{2}\int_0^t\int_0^{+\infty}\int_0^{+\infty}|E_n(s,z)|\,|\partial_x K(z,y)|\left(\frac{2}{\lambda}\,(z\wedge y)^\lambda\right)(\mu^n_s+\mu_s)\,(dy)dz\,ds \\ &\leq \frac{\kappa_1}{\lambda}\int_0^t\int_0^{+\infty}\int_0^{+\infty}|E_n(s,z)|z^{\lambda-1}y^\lambda\,(\mu^n_s+\mu_s)\,(dy)dz\,ds \\ &\leq \frac{\kappa_1}{\lambda}\int_0^t d_\lambda(\mu^n_s,\mu_s)\,M_\lambda(\mu^n_s+\mu_s)ds. \end{aligned} \tag{5.11}$$

We used (2.3).

Term $B_3(t)$.

Remark that $\left|\left(A\mathbb{1}_{(x,+\infty)}\right)(v,v)\right| = |\mathbb{1}_{(x,+\infty)}(2v) - 2\,\mathbb{1}_{(x,+\infty)}(v)| \leq 2\,\mathbb{1}_{\{v>\frac{x}{2}\}}$.

Since $\int_0^{+\infty} \mathbb{1}_{\{v>\frac{x}{2}\}} x^{\lambda-1}dx = \frac{(2v)^\lambda}{\lambda}$, we deduce:

$$\begin{aligned} B_3(t) &\le \frac{1}{2n}\int_0^{+\infty} x^{\lambda-1}\int_0^t\int_0^{+\infty} K(v,v)\left|\left(A\mathbb{1}_{(0,x]}\right)(v,v)\right|\mu_s^n(dv)\,dsdx \\ &\le \frac{\kappa_0}{\lambda n}\int_0^t ds\int_0^{+\infty}(2v)^{2\lambda}\mu_s^n(dv) \\ &\le \frac{2^{2\lambda}\kappa_0}{\lambda n}\int_0^t M_{2\lambda}(\mu_s^n)\,ds. \end{aligned} \tag{5.12}$$

We used (5.4) and (2.3).

Term $B_4(t)$.

First, remark that from (5.4) we have $\left|\varphi''_{\theta^n_{(x)}}(z)\right| \le \frac{2}{\theta^n_{(x)}}$ for all z, whence, due to the Taylor-Lagrange inequality,

$$\begin{aligned} \Bigg|\varphi_{\theta^n_{(x)}}\left(E_n(s,x)+\frac{1}{n}\left(A\mathbb{1}_{(x,+\infty)}\right)\left(X_s^i,X_s^j\right)\right) - \varphi_{\theta^n_{(x)}}(E_n(s,x)) \\ -\frac{1}{n}\left(A\mathbb{1}_{(x,+\infty)}\right)\left(X_s^i,X_s^j\right)\varphi'_{\theta^n_{(x)}}(E_n(s,x))\Bigg| \\ \le \frac{2}{\theta^n_{(x)}}\left[\frac{1}{n}\left(A\mathbb{1}_{(x,+\infty)}\right)(X_s^i,X_s^j)\right]^2. \end{aligned}$$

Then,

$$\begin{aligned} \mathbb{E}\left[B_4(t)\right] &\le \int_0^{+\infty} x^{\lambda-1}\mathbb{E}\Bigg[\int_0^t\int_{i<j}\int_0^{+\infty}\frac{2}{\theta^n_{(x)}}\left[\frac{1}{n}\left(A\mathbb{1}_{(x,+\infty)}\right)(X_{s-}^i,X_{s-}^j)\right]^2\mathbb{1}_{\{j\le N(s-)\}} \\ &\qquad \mathbb{1}_{\left\{z\le\frac{K\left(X_{s-}^i,X_{s-}^j\right)}{n}\right\}}J(ds,d(i,j),dz)\Bigg]dx \\ &\le \frac{2}{n}\int_0^t\mathbb{E}\left[\int_0^{+\infty}x^{\lambda-1}\sum_{i<j\le N(s)}\frac{K(X_s^i,X_s^j)}{n^2\,\theta^n_{(x)}}\left[\left(A\mathbb{1}_{(x,+\infty)}\right)(X_s^i,X_s^j)\right]^2dx\right]ds \\ &\le \frac{2\kappa_0}{n}\int_0^t\mathbb{E}\Bigg[\int_0^{+\infty}\frac{x^{\lambda-1}}{n^2\,\theta^n_{(x)}}\sum_{i<j\le N(s)}\left(X_s^i+X_s^j\right)^\lambda \\ &\qquad \left(\mathbb{1}_{x<X_s^i\wedge X_s^j}+\mathbb{1}_{X_s^i\vee X_s^j<x<X_s^i+X_s^j}\right)dx\Bigg]ds. \end{aligned}$$

We used (2.3) and (5.9) (since the sets are disjoint, the product of indicators vanishes). Therefore, using that $(v+y)^\lambda < v^\lambda + y^\lambda$ and a symmetry argument, we get

$$\mathbb{E}\left[B_4(t)\right] \le \frac{4\kappa_0}{n}\int_0^t\mathbb{E}\left[\left\langle\mu_s^n(dv)\mu_s^n(dy),v^\lambda\int_0^{+\infty}\frac{x^{\lambda-1}}{\theta^n_{(x)}}\left(\mathbb{1}_{x<v\wedge y}+\mathbb{1}_{v\vee y<x<v+y}\right)dx\right\rangle\right]ds.$$

According to Lemma 5.1–*(iii)*, and since $(v\wedge y)^\alpha(v\vee y)^\beta \leq v^\alpha y^\beta + y^\alpha v^\beta$ for $\alpha\geq 0$ and $\beta\geq 0$, we have

$$\begin{aligned}\left\langle \mu_s^n(dv)\mu_s^n(dy), v^\lambda \int_0^{+\infty} \frac{x^{\lambda-1}}{\theta^n_{(x)}}\left(\mathbb{1}_{x<v\wedge y} + \mathbb{1}_{v\vee y<x<v+y}\right)dx\right\rangle \leq \\ \frac{2\sqrt{n}}{\lambda}\left\langle \mu_s^n(dv)\mu_s^n(dy), v^\lambda y^\lambda\right\rangle \\ +\sqrt{n}\left(2^{2\lambda+\varepsilon}+\frac{1}{\lambda}\right)\left\langle \mu_s^n(dv)\mu_s^n(dy), v^{2\lambda}y^{2\lambda+\varepsilon}+y^{2\lambda}v^{2\lambda+\varepsilon}\right\rangle \mathbb{1}_{\lambda\in(0,1/2)} \\ +\sqrt{n}\left(2^{2\lambda+\varepsilon}+\frac{1}{\lambda}\right)\left\langle \mu_s^n(dv)\mu_s^n(dy), vy^{4\lambda+\varepsilon-1}+yv^{4\lambda+\varepsilon-1}\right\rangle 1_{\lambda\in[1/2,1]}.\end{aligned}$$

Finally, we deduce the bound:

$$\begin{aligned}\mathbb{E}\left[B_4(t)\right] \quad\leq\quad & \frac{8\kappa_0}{\lambda\sqrt{n}}\int_0^t \mathbb{E}\Big[\left[M_\lambda(\mu_s^n)\right]^2 + C\left[M_{2\lambda}(\mu_s^n)M_{2\lambda+\varepsilon}(\mu_s^n)\right]\mathbb{1}_{\lambda\in(0,1/2)} \\ & +C\left[M_1(\mu_s^n)M_{4\lambda+\varepsilon-1}(\mu_s^n)\right]\mathbb{1}_{\lambda\in[1/2,1]}\Big]ds,\end{aligned} \tag{5.13}$$

where $C=\left(\lambda 2^{2\lambda+\varepsilon}+1\right)$.

Conclusion.

Gathering (5.8), (5.11), (5.12) and (5.13), from (5.7), we get:

$$\begin{aligned}\mathbb{E}\left[d_\lambda(\mu_t^n,\mu_t)\right] \quad\leq\quad & \int_0^{+\infty} x^{\lambda-1}\varphi_{\theta^n_{(x)}}(E_n(0,x))dx \\ & +\frac{\kappa_0(\lambda+\varepsilon)}{\lambda\varepsilon\sqrt{n}}\int_0^t \mathbb{E}\left[M_0(\mu_s^n+\mu_s)+M_\lambda(\mu_s^n+\mu_s)\right]ds + \frac{\kappa_1}{\lambda}\int_0^t \mathbb{E}\left[d_\lambda(\mu_s^n,\mu_s)\,M_\lambda(\mu_s^n+\mu_s)\right]ds \\ & +\frac{2^{2\lambda}\kappa_0}{n\lambda}\int_0^t \mathbb{E}\left[M_{2\lambda}(\mu_s^n)\right]ds + \frac{8\kappa_0}{\lambda\sqrt{n}}\int_0^t \mathbb{E}\left[M_\lambda(\mu_s^n)\right]^2 ds \\ & +\frac{8C\kappa_0}{\lambda\sqrt{n}}\int_0^t \mathbb{E}\Big[\left[M_{2\lambda}(\mu_s^n)\,M_{2\lambda+\varepsilon}(\mu_s^n)\right]\mathbb{1}_{\lambda\in(0,1/2)} + \left[M_1(\mu_s^n)M_{4\lambda+\varepsilon-1}(\mu_s^n)\right]\mathbb{1}_{\lambda\in[1/2,1]}\Big]ds.\end{aligned}$$

We use (5.6) to bound the first term on the right-hand side. According to Proposition A.4 –*(a)*, $M_\alpha(\mu_s^n+\mu_s)\leq M_\alpha(\mu_0^n+\mu_0)$ a.s. for $\alpha\leq 1$. Since μ_0^n is deterministic, we get (recall that $2\lambda+\varepsilon<1$ if $\lambda\in(0,1/2)$):

$$\begin{aligned}\mathbb{E}\left[d_\lambda(\mu_t^n,\mu_t)\right] \leq d_\lambda(\mu_0^n,\mu_0) + \frac{2}{\lambda\sqrt{n}} + \frac{t\kappa_0(\lambda+\varepsilon)}{\lambda\varepsilon\sqrt{n}}\left(M_0(\mu_0^n+\mu_0)+M_\lambda(\mu_0^n+\mu_0)\right) \\ +\frac{\kappa_1}{\lambda}M_\lambda(\mu_0^n+\mu_0)\int_0^t \mathbb{E}\left[d_\lambda(\mu_s^n,\mu_s)\right]ds + \frac{2^{2\lambda}\kappa_0}{n\lambda}\int_0^t \mathbb{E}\left[M_{2\lambda}(\mu_s^n)\right]ds + \frac{8t\kappa_0}{\lambda\sqrt{n}}\left[M_\lambda(\mu_0^n)\right]^2 \\ +\frac{8Ct\kappa_0}{\lambda\sqrt{n}}\left[M_{2\lambda}(\mu_0^n)M_{2\lambda+\varepsilon}(\mu_0^n)\right]\mathbb{1}_{\lambda\in(0,1/2)} \\ +\frac{8C\kappa_0}{\lambda\sqrt{n}}M_1(\mu_0^n)\int_0^t \mathbb{E}\left[M_{4\lambda+\varepsilon-1}(\mu_s^n)\right]\mathbb{1}_{\lambda\in[1/2,1]}ds.\end{aligned}$$

Again, according to Proposition A.4 –*(b)*, $\mathbb{E}\left[M_\alpha(\mu_s^n)\right] \le M_\alpha(\mu_0^n)\exp[s\, C_{\lambda,\alpha} M_\lambda(\mu_0^n)]$ for $\alpha > 1$, and where $C_{\lambda,\alpha}$ is a positive constant depending on λ, α and κ_0. Thus

$$\begin{aligned}\mathbb{E}\left[d_\lambda(\mu_t^n,\mu_t)\right] \le{}& d_\lambda(\mu_0^n,\mu_0) + \frac{2}{\lambda\sqrt{n}} + \frac{t\kappa_0(\lambda+\varepsilon)}{\lambda\varepsilon\sqrt{n}}\left(M_0(\mu_0^n+\mu_0) + M_\lambda(\mu_0^n+\mu_0)\right)\\ &+\frac{\kappa_1}{\lambda}M_\lambda(\mu_0^n+\mu_0)\int_0^t \mathbb{E}\left[d_\lambda(\mu_s^n,\mu_s)\right]ds + \frac{2^{2\lambda}t\kappa_0}{n\lambda}M_{2\lambda}(\mu_0^n)\exp[t\, C_{\lambda,\varepsilon}M_\lambda(\mu_0^n)]\\ &+\frac{8t\kappa_0}{\lambda\sqrt{n}}\left[M_\lambda(\mu_0^n)\right]^2 + \frac{8Ct\kappa_0}{\lambda\sqrt{n}}\left[M_{2\lambda}(\mu_0^n)M_{2\lambda+\varepsilon}(\mu_0^n)\right]\mathbb{1}_{\lambda\in(0,1/2)}\\ &+\frac{8Ct\kappa_0}{\lambda\sqrt{n}}M_1(\mu_0^n)M_{4\lambda+\varepsilon-1}(\mu_0^n)\exp[t\, C_{\lambda,\varepsilon}M_\lambda(\mu_0^n)]\mathbb{1}_{\lambda\in[1/2,1]}.\end{aligned}$$

Recall that $\gamma = \max\{2\lambda, 4\lambda-1\}$. Observe that for $\mu \in \mathcal{M}^+$, $M_\alpha(\mu) \le M_0(\mu) + M_\beta(\mu)$ for any $0 \le \alpha \le \beta$. Elementary computations allow us to get:

$$\begin{aligned}\mathbb{E}\left[d_\lambda(\mu_t^n,\mu_t)\right] \quad\le\quad & d_\lambda(\mu_0^n,\mu_0) + (1+t)\frac{C_{\lambda,\varepsilon}}{\sqrt{n}}\left(1 + \left[M_0(\mu_0^n+\mu_0)\right]^2 + \left[M_{\gamma+\varepsilon}(\mu_0^n+\mu_0)\right]^2\right)\\ &\times\exp\left[t\, C_{\lambda,\varepsilon}M_\lambda(\mu_0^n+\mu_0)\right] + C_{\lambda,\varepsilon}M_\lambda(\mu_0^n+\mu_0)\int_0^t \mathbb{E}\left[d_\lambda(\mu_s^n,\mu_s)\right]ds,\end{aligned}$$

for some positive constant $C_{\lambda,\varepsilon}$ depending on λ, ε, κ_0 and κ_1. We conclude using the Gronwall lemma that Theorem 3.1 holds under (2.3).

6. Special Case

Now we are going to study the special case (2.4) for which $\lambda \in (0,1]$. We have a better result and a simpler proof than (2.3).

In the whole section, we assume that K satisfies (2.4) for some fixed $\lambda \in (0,1]$. We fix $\varepsilon > 0$, and we assume that $\mu_0 \in \mathcal{M}_\lambda^+ \cap \mathcal{M}_{2\lambda+\varepsilon}^+$. We denote by $(\mu_t)_{t\ge0}$ the unique (μ_0, K, λ)-weak solution to the Smoluchowski equation. We also consider the (n, K, μ_0^n)-Marcus Lushnikov process, for some given initial condition $\mu_0^n = \frac{1}{n}\sum_{i=1}^N \delta_{x_i}$.

As we did before we introduce $E_n(t,x) = F^{\mu_t^n}(x) - F^{\mu_t}(x)$ for $x \in (0,+\infty)$, as defined in (2.6). We observe that $\mathbb{1}_{(x,+\infty)} \in \mathcal{H}_\lambda^e$, since $\sup_{v>0} v^{-\lambda}|1_{(x,+\infty)}(v)| = x^{-\lambda} < +\infty$. Exactly as in Section 5 (see (5.3), take the absolute value and integrate against $x^{\lambda-1}dx$), we obtain:

(6.1) $$d_\lambda(\mu_t^n,\mu_t) \le d_\lambda(\mu_0^n,\mu_0) + C_1(t) + C_2(t) + C_3(t) + C_4(t),$$

where

$$
\begin{aligned}
C_1(t) &= \frac{1}{2}\int_0^t\int_0^{+\infty}\int_0^{+\infty} x^{\lambda-1}\Big[\mathbb{1}_{x>y}K(x-y,y)|E_n(s,x-y)| + |E_n(s,x)|K(x,y)\Big]\,dx \\
&\qquad (\mu_s^n+\mu_s)\,(dy)\,ds,\\
C_2(t) &= \frac{1}{2}\int_0^t\int_0^{+\infty}\int_0^{+\infty}\int_0^{+\infty} x^{\lambda-1}|\partial_x K(z,y)|\,\big|\big(A\mathbb{1}_{(x,+\infty)}\big)\,(z,y)\big|\,|E_n(s,z)|\,dz \\
&\qquad (\mu_s^n+\mu_s)\,(dy)\,ds,\\
C_3(t) &= \frac{1}{2n}\int_0^t\int_0^{+\infty}\int_0^{+\infty} x^{\lambda-1}K(v,v)\,\big|\mathbb{1}_{(x,+\infty)}(2v) - 2\,\mathbb{1}_{(x,+\infty)}(v)\big|\,dx\,\mu_s^n(dv)\,ds,\\
C_4(t) &= \int_0^{+\infty} x^{\lambda-1}\Bigg|\frac{1}{n}\int_0^t\int_{i<j}\int_0^{+\infty}(A\mathbb{1}_{(x,+\infty)})\left(X_{s-}^i, X_{s-}^j\right)\mathbb{1}_{\left\{z\leq\frac{K\left(X_{s-}^i,X_{s-}^j\right)}{n}\right\}}\mathbb{1}_{\{j\leq N(s-)\}}\\
&\qquad \tilde{J}(ds,d(i,j),dz)\Bigg|\,dx.
\end{aligned}
$$

We now study each term separately.

Term $C_1(t)$.

We have, using the change of variable $x\mapsto w+y$, (2.4) and using the fact that $x^{\lambda-1}$ is a non-increasing function:

$$
\begin{aligned}
C_1(t) &\leq \frac{\kappa_0}{2}\int_0^t\int_0^{+\infty}\int_0^{+\infty}(w+y)^{\lambda-1}(w\wedge y)^\lambda|E_n(s,w)|\,dw\,(\mu_s^n+\mu_s)\,(dy)\,ds\\
&\quad+\frac{\kappa_0}{2}\int_0^t\int_0^{+\infty}\int_0^{+\infty} x^{\lambda-1}(x\wedge y)^\lambda|E_n(s,x)|\,dx\,(\mu_s^n+\mu_s)\,(dy)\,ds\\
&\leq \frac{\kappa_0}{2}\int_0^t\int_0^{+\infty}\int_0^{+\infty} w^{\lambda-1}y^\lambda|E_n(s,w)|\,dw\,(\mu_s^n+\mu_s)\,(dy)\,ds\\
&\quad+\frac{\kappa_0}{2}\int_0^t\int_0^{+\infty}\int_0^{+\infty} x^{\lambda-1}y^\lambda|E_n(s,x)|\,dx\,(\mu_s^n+\mu_s)\,(dy)\,ds\\
&\leq \kappa_0\int_0^t M_\lambda(\mu_s^n+\mu_s)\,d_\lambda(\mu_s^n,\mu_s)ds. \qquad (6.2)
\end{aligned}
$$

Term $C_2(t)$.

Recall (5.10), use (2.4), we have immediately:

$$
\begin{aligned}
C_2(t) &\leq \frac{1}{\lambda}\int_0^t\int_0^{+\infty}\int_0^{+\infty}|\partial_x K(z,y)|\,(z\wedge y)^\lambda\,|E_n(s,z)|\,(\mu_s^n+\mu_s)\,(dy)\,ds\\
&\leq \frac{\kappa_1}{\lambda}\int_0^t\int_0^{+\infty}\int_0^{+\infty}|E_n(s,z)|z^{\lambda-1}y^\lambda\,(\mu_s^n+\mu_s)\,(dy)dz\,ds\\
&= \frac{\kappa_1}{\lambda}\int_0^t d_\lambda(\mu_s^n,\mu_s)\,M_\lambda(\mu_s^n+\mu_s)ds. \qquad (6.3)
\end{aligned}
$$

Term $C_3(t)$:

As before, recalling (5.12), we write:

$$C_3(t) \le \frac{2^{2\lambda}\kappa_0}{\lambda\, n}\int_0^t M_{2\lambda}(\mu_s^n)\,ds. \tag{6.4}$$

Term $C_4(t)$:

The submartingale term is going to be treated exactly as in the case $\lambda < 0$. Using similar arguments as for the term $A_3(t)$, we get

$$\mathbb{E}\left[\sup_{s\in[0,t]} C_4(s)\right] \le \frac{4}{\sqrt{n}}\int_0^{+\infty} x^{\lambda-1} \left\{\mathbb{E}\left[\int_0^t \left\langle \mu_s^n(dv)\mu_s^n(dy)\,,\, K(v,y)\ \left[\left(A\mathbb{1}_{(x,+\infty)}\right)(v,y)\right]^2\right\rangle\, ds\right]\right\}^{\frac{1}{2}} dx.$$

Using now (5.9) and (2.4), we deduce that

$$\mathbb{E}\left[\sup_{s\in[0,t]} C_4(s)\right] \le \frac{4\sqrt{\kappa_0}}{\sqrt{n}}\int_0^{+\infty} x^{\lambda-1}\left\{\mathbb{E}\left[\int_0^t \left\langle \mu_s^n(dv)\mu_s^n(dy)\,,\,(v\wedge y)^\lambda \left[\mathbb{1}_{\{x\in(0,v\wedge y)\}} + \mathbb{1}_{\{x\in(v\vee y,v+y)\}}\right]\right\rangle ds\right]\right\}^{\frac{1}{2}} dx. \tag{6.5}$$

First assume that $x \le 1$. Since $\mathbb{1}_{\{x\in(0,v\wedge y)\}} \le \dfrac{(v\wedge y)^\lambda}{x^\lambda}$, since $\mathbb{1}_{\{x\in(v\vee y,v+y)\}} \le \dfrac{(v+y)^\lambda}{x^\lambda} \le 2^\lambda\dfrac{(v\vee y)^\lambda}{x^\lambda}$, and since $(v\wedge y)^\lambda(v\wedge y)^\lambda \le v^\lambda y^\lambda$ and $(v\wedge y)^\lambda(v\vee y)^\lambda = v^\lambda y^\lambda$, we deduce that

$$\left\langle \mu_s^n(dv)\mu_s^n(dy)\,,(v\wedge y)^\lambda\left[\mathbb{1}_{\{x\in(0,v\wedge y)\}} + \mathbb{1}_{\{x\in(v\vee y,v+y)\}}\right]\right\rangle \le \frac{\left(1+2^\lambda\right)}{x^\lambda}[M_\lambda(\mu_s^n)]^2.$$

Thus,

$$\begin{aligned}
&\int_0^1 x^{\lambda-1}\left\{\mathbb{E}\left[\int_0^t \langle \mu_s^n(dv)\mu_s^n(dy)\,,\,(v\wedge y)^\lambda\left[\mathbb{1}_{\{x\in(0,v\wedge y)\}} + \mathbb{1}_{\{x\in(v\vee y,v+y)\}}\right]\rangle\, ds\right]\right\}^{\frac{1}{2}} dx\\
&\qquad \le \sqrt{1+2^\lambda}\int_0^1 x^{\frac{\lambda}{2}-1}dx \times \left\{\mathbb{E}\left[\int_0^t [M_\lambda(\mu_s^n)]^2\,ds\right]\right\}^{\frac{1}{2}}\\
&\qquad = \frac{2\sqrt{1+2^\lambda}}{\lambda}\left\{\mathbb{E}\left[\int_0^t [M_\lambda(\mu_s^n)]^2\,ds\right]\right\}^{\frac{1}{2}}.
\end{aligned} \tag{6.6}$$

Next consider $x > 1$. Since $\mathbb{1}_{\{x\in(0,v\wedge y)\}} \le \dfrac{(v\wedge y)^{2\lambda+\varepsilon}}{x^{2\lambda+\varepsilon}}$, and $\mathbb{1}_{\{x\in(v\vee y,v+y)\}} \le \dfrac{(v+y)^{2\lambda+\varepsilon}}{x^{2\lambda+\varepsilon}} \le 2^{2\lambda+\varepsilon}\dfrac{(v\vee y)^{2\lambda+\varepsilon}}{x^{2\lambda+\varepsilon}}$, and since $(v\wedge y)^\lambda(v\wedge y)^{2\lambda+\varepsilon} \le v^\lambda y^{2\lambda+\varepsilon}$ and $(v\wedge y)^\lambda(v\vee y)^{2\lambda+\varepsilon} \le v^\lambda y^{2\lambda+\varepsilon} + v^{2\lambda+\varepsilon}y^\lambda$, and using the symmetry, we deduce that

$$\left\langle \mu_s^n(dv)\mu_s^n(dy)\,,(v\wedge y)^\lambda\left[\mathbb{1}_{\{x\in(0,v\wedge y)\}} + \mathbb{1}_{\{x\in(v\vee y,v+y)\}}\right]\right\rangle \le \frac{\left(1+2^{2\lambda+\varepsilon+1}\right)}{x^{2\lambda+\varepsilon}}M_\lambda(\mu_s^n)M_{2\lambda+\varepsilon}(\mu_s^n).$$

Thus,

$$
\begin{aligned}
&\int_1^{+\infty} x^{\lambda-1}\left\{\mathbb{E}\left[\int_0^t \left\langle \mu_s^n(dv)\mu_s^n(dy)\,,\,(v\wedge y)^\lambda\left[\mathbb{1}_{\{x\in(0,v\wedge y)\}}+\mathbb{1}_{\{x\in(v\vee y,v+y)\}}\right]\right\rangle ds\right]\right\}^{\frac{1}{2}} dx\\
&\le \sqrt{1+2^{2\lambda+\varepsilon+1}}\int_1^{+\infty} x^{-\frac{\varepsilon}{2}-1}dx\times\left\{\mathbb{E}\left[\int_0^t M_\lambda(\mu_s^n)M_{2\lambda+\varepsilon}(\mu_s^n)\,ds\right]\right\}^{\frac{1}{2}} \qquad (6.7)\\
&= \frac{2\sqrt{1+2^{2\lambda+\varepsilon+1}}}{\varepsilon}\left\{\mathbb{E}\left[\int_0^t M_\lambda(\mu_s^n)M_{2\lambda+\varepsilon}(\mu_s^n)\,ds\right]\right\}^{\frac{1}{2}}.
\end{aligned}
$$

Gathering (6.5), (6.6) and (6.7), we obtain:

$$
\begin{aligned}
\mathbb{E}\left[\sup_{s\in[0,t]} C_4(s)\right] \le\ & \frac{8\sqrt{\kappa_0}}{\sqrt{n}}\left\{\frac{\sqrt{1+2^\lambda}}{\lambda}\left(\mathbb{E}\left[\int_0^t [M_\lambda(\mu_s^n)]^2\,ds\right]\right)^{\frac{1}{2}}\right.\\
&\left.+\frac{\sqrt{1+2^{2\lambda+\varepsilon+1}}}{\varepsilon}\left(\mathbb{E}\left[\int_0^t M_\lambda(\mu_s^n)M_{2\lambda+\varepsilon}(\mu_s^n)\,ds\right]\right)^{\frac{1}{2}}\right\}. \qquad (6.8)
\end{aligned}
$$

<u>**Conclusion.**</u>

Therefore, gathering (6.2), (6.3), (6.4) and (6.8), we obtain:

$$
\begin{aligned}
\mathbb{E}\left[\sup_{s\in[0,t]} d_\lambda(\mu_s^n,\mu_s)\right] \le\ & d_\lambda(\mu_0^n,\mu_0)+\left(\kappa_0+\frac{\kappa_1}{\lambda}\right)\int_0^t \mathbb{E}\left[d_\lambda(\mu_s^n,\mu_s)\,M_\lambda(\mu_s^n+\mu_s)\right]ds\\
&+\frac{2^{2\lambda}\kappa_0}{n\,\lambda}\int_0^t \mathbb{E}\left[M_{2\lambda}(\mu_s^n)\right]\,ds\\
&+\frac{8\sqrt{\kappa_0}}{\sqrt{n}}\left\{\frac{\sqrt{1+2^\lambda}}{\lambda}\left(\mathbb{E}\left[\int_0^t [M_\lambda(\mu_s^n)]^2\,ds\right]\right)^{\frac{1}{2}}\right.\\
&\left.+\frac{\sqrt{1+2^{2\lambda+\varepsilon+1}}}{\varepsilon}\left(\mathbb{E}\left[\int_0^t M_\lambda(\mu_s^n)M_{2\lambda+\varepsilon}(\mu_s^n)\,ds\right]\right)^{\frac{1}{2}}\right\}.
\end{aligned}
$$

Observe that $M_\alpha(\mu_0^n) \le M_0(\mu_0^n)+M_{2\lambda+\varepsilon}(\mu_0^n)$ for $\alpha=\lambda,\,2\lambda$. Proposition A.4 implies that for $\alpha\in(0,1]$, $M_\alpha(\mu_t^n+\mu_t)\le M_\alpha(\mu_0^n+\mu_0)$ a.s. and for $\alpha=2\lambda,\,2\lambda+\varepsilon$, $\mathbb{E}\left[M_\alpha(\mu_s^n)\right]\le M_\alpha(\mu_0^n)\exp[s\,C_{\lambda,\alpha}M_\lambda(\mu_0^n)]$ where $C_{\lambda,\alpha}$ is a positive constant depending on λ, α, κ_0 and κ_1. Since μ_0^n is deterministic, we deduce that

$$
\begin{aligned}
\mathbb{E}\left[\sup_{s\in[0,t]} d_\lambda(\mu_s^n,\mu_s)\right] \le\ & d_\lambda(\mu_0^n,\mu_0)+(1+t)\frac{C_{\lambda,\varepsilon}}{\sqrt{n}}\left(M_0(\mu_0^n)+M_{2\lambda+\varepsilon}(\mu_0^n)\right)\exp[t\,C_{\lambda,\varepsilon}M_\lambda(\mu_0^n)]\\
&+C_{\lambda,\varepsilon}\,M_\lambda(\mu_0^n+\mu_0)\int_0^t \mathbb{E}\left[d_\lambda(\mu_s^n,\mu_s)\right]ds,
\end{aligned}
$$

for some positive constant $C_{\lambda,\varepsilon}$ depending on λ, ε, κ_0 and κ_1. We conclude using the Gronwall lemma.

7. Choice of the initial condition

The aim of this section is to prove Proposition 3.2. We thus fix $\lambda\in(-\infty,1]\setminus\{0\}$ and $\mu_0\in\mathcal{M}_\lambda^+\cap M_{2\lambda}^+$. We first treat the case where μ_0 is atomless, next the case where μ_0 is discrete.

7.1. Continuum System. — We assume that μ_0 is atomless. For $0 < a < A < +\infty$, we consider $\mu_0|_K$, the restriction of μ_0 to $K = [a, A]$. We consider also N points $a = x_0 < x_1 < \cdots < x_N \le A$ such that:

$$\mu_0\left([x_{i-1},\, x_i)\right) = \frac{1}{n}, \quad \forall\, i = 1, \cdots, N \qquad \text{and} \qquad \mu_0\left([x_N,\, A]\right) < \frac{1}{n}. \tag{7.1}$$

We will use the points $\{x_i\}_{i=1,\cdots,N}$ to construct the discrete measure μ_0^n choosing a and A following the value of λ as a function of n.

7.1.1. Case $\lambda \in (-\infty, 0)$:— First, we choose $a_n < A_n$ as follows:

$$a_n = \left(\frac{1}{\sqrt{n}}\right)^{\frac{1}{|\lambda|}} \qquad \text{and} \qquad \int_{A_n}^{+\infty} x^\lambda \mu_0(dx) \le \frac{1}{\sqrt{n}}. \tag{7.2}$$

Next, we assign the weight $\mu_0\left([x_{i-1},\, x_i)\right) = \frac{1}{n}$ to the point x_i and we set

$$\mu_0^n = \frac{1}{n}\sum_{i=1}^{N_n} \delta_{x_i}. \tag{7.3}$$

If $\alpha \le 0$, we get:

$$\begin{aligned} M_\alpha(\mu_0^n) &= \frac{1}{n}\sum_{i=1}^{N_n} x_i^\alpha = \sum_{i=1}^{N_n} x_i^\alpha\, \mu_0([x_{i-1},\, x_i)) = \sum_{i=1}^{N_n}\int_0^{+\infty} x_i^\alpha\, \mathbb{1}_{[x_{i-1},\, x_i)}(x)\mu_0(dx) \\ &\le \sum_{i=1}^{N_n}\int_0^{+\infty} x^\alpha\, \mathbb{1}_{[x_{i-1},\, x_i)}(x)\mu_0(dx) = \int_{a_n}^{x_{N_n}} x^\alpha \mu_0(dx) \le M_\alpha(\mu_0). \end{aligned} \tag{7.4}$$

For the distance, we have, with $K_n = [a_n, A_n]$:

$$\begin{aligned} d_\lambda(\mu_0|_{K_n}, \mu_0) &= \int_0^{+\infty} x^{\lambda-1}\left|\int_0^{+\infty} \mathbb{1}_{(0,x)}(y)\left(\mu_0|_{K_n} - \mu_0\right)(dy)\right| dx \\ &= \int_0^{+\infty} x^{\lambda-1}\left[\mu_0\left((0,x)\right)\mathbb{1}_{x<a_n} + \mu_0\left((A_n,x)\right)\mathbb{1}_{x>A_n} + \mu_0\left((0,a_n)\right)\mathbb{1}_{x>a_n}\right] dx \\ &= \int_0^{a_n}\int_y^{a_n} x^{\lambda-1}\, dx\, \mu_0(dy) + \int_{A_n}^{+\infty}\int_y^{+\infty} x^{\lambda-1}\, dx\, \mu_0(dy) \\ &\quad + \int_0^{a_n}\int_{a_n}^{+\infty} x^{\lambda-1}\, dx\, \mu_0(dy) \\ &\le 2\int_0^{a_n}\int_y^{+\infty} x^{\lambda-1}\, dx\, \mu_0(dy) + \int_{A_n}^{+\infty}\int_y^{+\infty} x^{\lambda-1}\, dx\, \mu_0(dy) \\ &\le \frac{2a_n^{|\lambda|}}{|\lambda|}\int_0^{+\infty} y^{2\lambda}\mu_0(dy) + \frac{1}{|\lambda|}\int_{A_n}^{+\infty} y^\lambda \mu_0(dy) \le \frac{1}{|\lambda|\sqrt{n}}\left(2M_{2\lambda}(\mu_0) + 1\right), \end{aligned} \tag{7.5}$$

we used (7.2) for the last inequality. Next, we introduce the notation $i_x = \max\{i : x_i \le x;\, i = 0, \cdots, N_n\}$ for $x > a_n$. We remark that $\mu_0^n\left((0,x]\right) = 0$ if $x \le a_n$ and $\mu_0^n\left((0,x]\right) = \mu_0\left((a_n, x_{i_x}]\right)$ if

$x > a_n$. Hence,

$$\begin{aligned} d_\lambda(\mu_0^n, \mu_0|_{K_n}) &= \int_0^{+\infty} x^{\lambda-1} \left| \int_0^{+\infty} \mathbb{1}_{(0,x)}(y) \, (\mu_0^n - \mu_0|_{K_n}) \, (dy) \right| dx \\ &= \int_{a_n}^{A_n} x^{\lambda-1} \left| \mu_0([a_n, x_{i_x})) - \mu_0([a_n, x)) \right| dx \\ &\quad + \int_{A_n}^{+\infty} x^{\lambda-1} \left| \mu_0([a_n, x_{i_x})) - \mu_0([a_n, A_n)) \right| dx \\ &\le \int_{a_n}^{A_n} x^{\lambda-1} \mu_0((x_{i_x}, x)) \, dx + \int_{A_n}^{+\infty} x^{\lambda-1} \mu_0([x_{N_n}, A_n)) \, dx \\ &\le \frac{2}{n} \int_{a_n}^{+\infty} x^{\lambda-1} dx = \frac{2}{|\lambda| n} a_n^\lambda \le \frac{2}{|\lambda| \sqrt{n}} \, . \end{aligned} \tag{7.6}$$

We used $|\mu_0([a_n, x_{i_x})) - \mu_0([a_n, x))| = \mu_0((x_{i_x}, x)) \le \mu_0([x_{j-1}, x_j)) \le \frac{1}{n}$ for some $j = 1, \cdots, N$, and (7.2). Finally, from (7.5) and (7.6), we obtain:

$$d_\lambda(\mu_0^n, \mu_0) \le d_\lambda(\mu_0^n, \mu_0|_{K_n}) + d_\lambda(\mu_0|_{K_n}, \mu_0) \le \frac{1}{|\lambda| \sqrt{n}} \left(2M_{2\lambda}(\mu_0) + 3 \right).$$

7.1.2. Case $\lambda \in (0, 1]$:— First, we choose $a_n < A_n$ as follows:

$$\int_0^{a_n} x^\lambda \mu_0(dx) \le \frac{1}{\sqrt{n}} \qquad \text{and} \qquad A_n = \left(\sqrt{n}\right)^{\frac{1}{\lambda}} . \tag{7.7}$$

Next, we assign the weight $\mu_0([x_{i-1}, x_i)) = \frac{1}{n}$ to the point x_{i-1}, recall that $x_0 = a_n$. We set

$$\mu_0^n(dx) = \frac{1}{n} \sum_{i=0}^{N_n - 1} \delta_{x_i} . \tag{7.8}$$

If $\alpha \ge 0$, we get:

$$\begin{aligned} M_\alpha(\mu_0^n) &= \frac{1}{n} \sum_{i=0}^{N_n-1} x_i^\alpha = \sum_{i=1}^{N_n} x_{i-1}^\alpha \, \mu_0([x_{i-1}, \, x_i)) = \sum_{i=1}^{N_n} \int_0^{+\infty} x_{i-1}^\alpha \, \mathbb{1}_{[x_{i-1}, \, x_i)}(x) \mu_0(dx) \\ &\le \sum_{i=1}^{N_n} \int_0^{+\infty} x^\alpha \, \mathbb{1}_{[x_{i-1}, \, x_i)}(x) \mu_0(dx) = \int_{a_n}^{x_{N_n}} x^\alpha \mu_0(dx) \le M_\alpha(\mu_0). \end{aligned} \tag{7.9}$$

For the distance, we have, with $K_n = [a_n, A_n]$:

$$\begin{aligned}
d_\lambda(\mu_0|_{K_n}, \mu_0) &= \int_0^{+\infty} x^{\lambda-1} \left| \int_0^{+\infty} \mathbb{1}_{[x,+\infty)}(y)\,(\mu_0|_{K_n} - \mu_0)\,(dy) \right| dx \\
&= \int_0^{+\infty} x^{\lambda-1} \left[\mu_0\left([x, a_n)\right) \mathbb{1}_{x<a_n} + \mu_0\left([x,+\infty)\right) \mathbb{1}_{x>A_n} + \mu_0\left([A_n,+\infty)\right) \mathbb{1}_{x<A_n} \right] dx \\
&= \int_0^{a_n} \int_0^{y} x^{\lambda-1}\, dx\, \mu_0(dy) + \int_{A_n}^{+\infty} \int_{A_n}^{y} x^{\lambda-1}\, dx\, \mu_0(dy) \\
&\quad + \int_{A_n}^{+\infty} \int_0^{A_n} x^{\lambda-1}\, dx\, \mu_0(dy) \\
&\leq \int_0^{a_n} \int_0^{y} x^{\lambda-1}\, dx\, \mu_0(dy) + 2 \int_{A_n}^{+\infty} \int_0^{y} x^{\lambda-1}\, dx\, \mu_0(dy) \\
&\leq \frac{1}{\lambda} \int_0^{a_n} x^{\lambda} \mu_0(dx) + \frac{2A_n^{-\lambda}}{\lambda} \int_0^{+\infty} y^{2\lambda} \mu_0(dy) \leq \frac{1}{\lambda\sqrt{n}} \left(1 + 2M_{2\lambda}(\mu_0)\right), \qquad (7.10)
\end{aligned}$$

we used (7.7) for the last inequality. Next, using the notation $i_x = \min\{i : x_i \geq x;\ i = 0, \cdots, N-1\}$ for $x > a_n$, we remark that $\mu_0^n\left([x,+\infty)\right) = 0$ if $x \geq A_n$ and $\mu_0^n\left([x,+\infty)\right) = \mu_0\left([x_{i_x}, A_n)\right)$ if $x < A_n$. Hence,

$$\begin{aligned}
d_\lambda(\mu_0^n, \mu_0|_{K_n}) &= \int_0^{+\infty} x^{\lambda-1} \left| \int_0^{+\infty} \mathbb{1}_{(x,+\infty)}(y)\,(\mu_0^n - \mu_0|_{K_n})\,(dy) \right| dx \\
&= \int_{a_n}^{A_n} x^{\lambda-1} \left| \mu_0((x_{i_x}, A_n)) - \mu_0((x, A_n)) \right| dx \\
&\quad + \int_0^{a_n} x^{\lambda-1} \left| \mu_0([x_{i_x}, A_n)) - \mu_0((a_n, A_n)) \right| dx \\
&= \int_{a_n}^{A_n} x^{\lambda-1} \mu_0((x, x_{i_x})) dx \;\leq\; \frac{1}{n} \int_0^{A_n} x^{\lambda-1} dx = \frac{1}{\lambda\, n} A_n^{\lambda} \leq \frac{1}{\lambda\sqrt{n}}, \qquad (7.11)
\end{aligned}$$

we used $|\mu_0((x_{i_x}, A_n)) - \mu_0((x, A_n))| = \mu_0((x, x_{i_x})) \leq \mu_0([x_{j-1}, x_j))$ for some $j = 1, \cdots, N$, and (7.7). Finally, from (7.10) and (7.11), we deduce:

$$d_\lambda(\mu_0^n, \mu_0) \quad\leq\quad \frac{2}{\lambda\sqrt{n}} \left(M_{2\lambda}(\mu_0) + 1\right).$$

7.2. Discrete System. — Let us thus, consider $\mu_0 \in \mathcal{M}^+$ with support in $\mathbb{N}$, i.e.

$$\mu_0 = \sum_{k\geq 1} \alpha_k\, \delta_k, \qquad \text{with } \alpha_k \in \mathbb{R}_+. \qquad (7.12)$$

We set for $A \in \mathbb{N}^*$:

$$\mu_0^A = \sum_{k=1}^{A} \alpha_k\, \delta_k. \qquad (7.13)$$

7.2.1. Case $\lambda \in (-\infty, 0)$:— We choose A_n such that:

$$\sum_{k>A_n} \alpha_k k^{\lambda} \leq \frac{1}{\sqrt{n}}, \qquad (7.14)$$

and we set,

$$\mu_0^n = \frac{1}{n}\sum_{k=1}^{A_n} \alpha_k^n\,\delta_k, \qquad \text{with} \tag{7.15}$$

$$\begin{cases} \alpha_1^n &= \lfloor n\alpha_1 \rfloor, \\ \alpha_k^n &= \lfloor n(\alpha_1+\cdots+\alpha_k)\rfloor - \lfloor n(\alpha_1+\cdots+\alpha_{k-1})\rfloor \text{ for } k=2,\cdots, A_n, \end{cases} \tag{7.16}$$

where $\lfloor\cdot\rfloor$ is the floor function. Remark that chosen in this way, the α_k^n are non-negative integers and μ_0^n can be written as $\frac{1}{n}\sum_{i=1}^{N_n}\delta_{x_i}$, hence μ_0^n is the measure we search. Observe that for $k=1,\cdots, A_n$, we have

$$\begin{aligned} \left|\sum_{i=1}^{k}\left(\frac{1}{n}\alpha_i^n - \alpha_i\right)\right| &= \left|\frac{1}{n}\left(\alpha_1^n+\cdots+\alpha_k^n\right) - (\alpha_1+\cdots+\alpha_k)\right| \\ &= \left|\frac{1}{n}\left\lfloor n\left(\alpha_1+\cdots+\alpha_k\right)\right\rfloor - (\alpha_1+\cdots+\alpha_k)\right| \le \frac{1}{n}. \end{aligned} \tag{7.17}$$

If $\alpha\le 0$, we have:

$$\begin{aligned} M_\alpha\left(\mu_0^n\right) &= \frac{1}{n}\sum_{k=1}^{A_n}\alpha_k^n k^\alpha \\ &= \frac{1}{n}\lfloor n\alpha_1\rfloor + \frac{1}{n}\sum_{k=2}^{A_n}\lfloor n(\alpha_1+\cdots+\alpha_k)\rfloor k^\alpha - \frac{1}{n}\sum_{k=2}^{A_n}\lfloor n(\alpha_1+\cdots+\alpha_{k-1})\rfloor k^\alpha \\ &= \frac{1}{n}\lfloor n\alpha_1\rfloor + \frac{1}{n}\sum_{k=2}^{A_n}\lfloor n(\alpha_1+\cdots+\alpha_k)\rfloor k^\alpha - \frac{1}{n}\sum_{k=1}^{A_n-1}\lfloor n(\alpha_1+\cdots+\alpha_k)\rfloor (k+1)^\alpha \\ &= \frac{1}{n}\left(\lfloor n\alpha_1\rfloor + A_n^\alpha\lfloor n(\alpha_1+\cdots+\alpha_{A_n})\rfloor - 2^\alpha\lfloor n\alpha_1\rfloor\right) \\ &\quad + \frac{1}{n}\sum_{k=2}^{A_n-1}\lfloor n(\alpha_1+\cdots+\alpha_k)\rfloor\left(k^\alpha-(k+1)^\alpha\right) \\ &\le \alpha_1\left(1-2^\alpha\right) + A_n^\alpha\left(\alpha_1+\cdots+\alpha_{A_n}\right) + \sum_{k=2}^{A_n-1}\left(\alpha_1+\cdots+\alpha_k\right)\left(k^\alpha-(k+1)^\alpha\right) \\ &= \sum_{k=1}^{A_n-1}\alpha_k\left[A_n^\alpha + \sum_{j=k}^{A_n-1}\left(j^\alpha-(j+1)^\alpha\right)\right] + A_n^\alpha\alpha_{A_n} = \sum_{k=1}^{A_n}\alpha_k k^\alpha \le M_\alpha(\mu_0). \end{aligned} \tag{7.18}$$

Next, for the distance, we have:

$$\begin{aligned} d_\lambda\left(\mu_0^{A_n},\mu_0\right) &\le \int_0^{+\infty} x^{\lambda-1}\left|\int_0^{+\infty}\mathbb{1}_{(0,x)}(y)\left(\mu_0^{A_n}-\mu_0\right)(dy)\right| dx \\ &= \int_0^{+\infty} x^{\lambda-1}\int_0^x \sum_{k>A_n}\alpha_k\,\delta_k(dy)dx = \sum_{k>A_n}\alpha_k\int_k^{+\infty}x^{\lambda-1}dx \\ &= \frac{1}{|\lambda|}\sum_{k>A_n}\alpha_k\,k^\lambda \le \frac{1}{|\lambda|\sqrt{n}}, \end{aligned} \tag{7.19}$$

we used (7.14). Next,

$$\begin{aligned} d_\lambda\left(\mu_0^n, \mu_0^{A_n}\right) &= \int_0^{+\infty} x^{\lambda-1}\left|\int_0^{+\infty} \mathbb{1}_{(0,x)}(y)\left(\mu_0^n - \mu_0^{A_n}\right)(dy)\right| dx \\ &= \sum_{k=1}^{A_n-1}\int_k^{k+1} x^{\lambda-1}\left|\sum_{i=1}^{k}\left(\frac{1}{n}\alpha_i^n - \alpha_i\right)\right| dx + \int_{A_n}^{+\infty} x^{\lambda-1}\left|\sum_{i=1}^{A_n}\left(\frac{1}{n}\alpha_i^n - \alpha_i\right)\right| dx \\ &\le \frac{2}{n}\int_1^{+\infty} x^{\lambda-1}dx \le \frac{2}{|\lambda| n}, \end{aligned} \tag{7.20}$$

we used (7.17) for the last inequality. Finally, from (7.19) and (7.20), we have:

$$d_\lambda\left(\mu_0^n, \mu_0\right) \le d_\lambda\left(\mu_0^n, \mu_0^{A_n}\right) + d_\lambda\left(\mu_0^{A_n}, \mu_0\right) \le \frac{1}{|\lambda|\sqrt{n}}\left(1 + \frac{2}{\sqrt{n}}\right).$$

7.2.2. Case $\lambda \in (0,1]$:— We set

$$A_n = \left\lfloor \left(\sqrt{n}\right)^{\frac{1}{\lambda}} \right\rfloor + 1, \tag{7.21}$$

Note that chosen in this way, we have $A_n^{-\lambda} \le \frac{1}{\sqrt{n}}$, implying

$$\sum_{k\ge A_n} \alpha_k\, k^\lambda \le A_n^{-\lambda}\sum_{k\ge A_n}\alpha_k\, k^{2\lambda} \le \frac{1}{\sqrt{n}} M_{2\lambda}(\mu_0). \tag{7.22}$$

We set the measure μ_0^n as defined in (7.15), with

$$\alpha_k^n = \left\lfloor n\sum_{i\ge k}\alpha_i\right\rfloor - \left\lfloor n\sum_{i\ge k+1}\alpha_i\right\rfloor, \quad \text{for } k = 1, \cdots, A_n. \tag{7.23}$$

Observe that, since $\sum_{k\ge1}\alpha_k = M_0(\mu_0) \le M_\lambda(\mu_0) = \sum_{k\ge1}\alpha_k k^\lambda < +\infty$, the weights $\{\alpha_k^n\}_{k\ge1}$ are well-defined. Remark that chosen in this way, the α_k^n are non-negative integers and μ_0^n can be written as $\frac{1}{n}\sum_{i=1}^{N_n}\delta_{x_i}$, hence μ_0^n is the measure we search.

For $1 \le j \le A_n$, we have:

$$\begin{aligned} \left|\sum_{k=j}^{A_n}\left(\frac{1}{n}\alpha_k^n - \alpha_k\right)\right| &= \left|\frac{1}{n}\left\lfloor n\sum_{i\ge j}\alpha_i\right\rfloor - \frac{1}{n}\left\lfloor n\sum_{i\ge A_n+1}\alpha_i\right\rfloor - \sum_{k=j}^{A_n}\alpha_k\right| \\ &\le \left|\frac{1}{n}\left\lfloor n\sum_{i=j}^{A_n}\alpha_i\right\rfloor + \frac{1}{n} - \sum_{k=j}^{A_n}\alpha_k\right| \\ &\le \frac{1}{n} + \left|\frac{1}{n}\left\lfloor n\sum_{i=j}^{A_n}\alpha_i\right\rfloor - \sum_{k=j}^{A_n}\alpha_k\right| \le \frac{2}{n}. \end{aligned} \tag{7.24}$$

If $\alpha \geq 0$, we have:

$$\begin{aligned} M_\alpha(\mu_0^n) &= \frac{1}{n}\sum_{k=1}^{A_n} \alpha_k^n k^\alpha = \frac{1}{n}\sum_{k=1}^{A_n}\left\lfloor n\sum_{i\geq k}\alpha_i\right\rfloor k^\alpha - \frac{1}{n}\sum_{k=1}^{A_n}\left\lfloor n\sum_{i\geq k+1}\alpha_i\right\rfloor k^\alpha \\ &= \frac{1}{n}\sum_{k=2}^{A_n}\left\lfloor n\sum_{i\geq k}\alpha_i\right\rfloor [k^\alpha-(k-1)^\alpha] + \frac{1}{n}\left\lfloor n\sum_{i\geq 1}\alpha_i\right\rfloor - \frac{A_n^\alpha}{n}\left\lfloor n\sum_{i\geq A_n+1}\alpha_i\right\rfloor \\ &\leq \sum_{k\geq 1}\left(\sum_{i\geq k}\alpha_i\right)[k^\alpha-(k-1)^\alpha] \;=\; \sum_{i\geq 1}\alpha_i\sum_{k=1}^{i}[k^\alpha-(k-1)^\alpha] \;=\; M_\alpha(\mu_0), \end{aligned} \tag{7.25}$$

For the distance, we have :

$$\begin{aligned} d_\lambda\left(\mu_0^{A_n},\mu_0\right) &= \int_0^{+\infty} x^{\lambda-1}\left|\int_0^{+\infty}\mathbb{1}_{[x,+\infty)}(y)\left(\mu_0^{A_n}-\mu_0\right)(dy)\right| dx \\ &= \int_0^{+\infty} x^{\lambda-1}\int_x^{+\infty}\sum_{k>A_n}\alpha_k\,\delta_k(dy)dx \;=\; \sum_{k>A_n}\alpha_k\int_0^k x^{\lambda-1}dx \\ &= \frac{1}{\lambda}\sum_{k>A_n}\alpha_k\,k^\lambda \;\leq\; \frac{1}{\lambda\sqrt{n}}M_{2\lambda}(\mu_0), \end{aligned} \tag{7.26}$$

we used (7.22). Next,

$$\begin{aligned} d_\lambda\left(\mu_0^n,\mu_0^{A_n}\right) &= \int_0^{+\infty} x^{\lambda-1}\left|\int_0^{+\infty}\mathbb{1}_{[x,+\infty)}(y)\left(\mu_0^n-\mu_0^{A_n}\right)(dy)\right| dx \\ &= \sum_{j=1}^{A_n}\int_{j-1}^{j} x^{\lambda-1}\left|\sum_{k=j}^{A_n}\left(\frac{1}{n}\alpha_k^n-\alpha_k\right)\right| dx \\ &\leq \frac{2}{n}\int_0^{A_n} x^{\lambda-1}dx \;\leq\; \frac{2A_n^\lambda}{\lambda\, n} \;\leq\; \frac{4}{\lambda\sqrt{n}}, \end{aligned} \tag{7.27}$$

we used (7.24) and (7.21). Finally, from (7.26) and (7.27), we obtain:

$$d_\lambda(\mu_0^n,\mu_0) \;\leq\; d_\lambda(\mu_0^n,\mu_0^{A_n}) + d_\lambda(\mu_0^{A_n},\mu_0) \;\leq\; \frac{1}{\lambda\sqrt{n}}\left(M_{2\lambda}(\mu_0)+4\right).$$

7.3. Conclusion. — In any case, ($\lambda \in (-\infty,1]\setminus\{0\}$ and μ_0 either atomless or with support in $\mathbb{N}$), we have built a measure of the form $\mu_0^n = \sum_{i=1}^{N_n}\delta_{x_i}$ satisfying the desired conditions on the moments and distance. It is straightforward to show that $N_n = n\,\langle\mu_0^n(dx),1\rangle$. Hence, according to (7.4), (7.9), (7.18) and (7.25), we deduce,

$$N_n = nM_0(\mu_0^n) \leq nM_0(\mu_0). \tag{7.28}$$

This concludes the proof of Proposition 3.2.

Appendix A

This section is devoted to some technical issues.

Lemma A.1. — *Consider $\lambda \in (-\infty, 1] \setminus \{0\}$. Then, there exists a positive constant C_ϕ depending on ϕ and λ such that*

$$\text{(A.1)} \quad \begin{cases} \text{if } \lambda \in (-\infty, 0), & (x+y)^\lambda |(A\phi)(x,y)| \leq C_\phi (xy)^\lambda \quad \forall \phi \in \mathcal{H}_\lambda, \\ \text{if } \lambda \in (0,1], & (x+y)^\lambda |(A\phi)(x,y)| \leq C_\phi \left(1 + x^{2\lambda} + y^{2\lambda}\right) \quad \forall \phi \in \mathcal{H}_\lambda, \\ \text{if } \lambda \in (0,1], & (x \wedge y)^\lambda \ |(A\phi)(x,y)| \leq C_\phi (xy)^\lambda \quad \forall \phi \in \mathcal{H}^e_\lambda. \end{cases}$$

Proof. — Assume first that $\lambda \in (-\infty, 0)$ and $\phi \in \mathcal{H}_\lambda$. Since $|\phi(x)| \leq Cx^\lambda$ for some constant $C > 0$, we have

$$(x+y)^\lambda |(A\phi)(x,y)| \leq C(x^\lambda \wedge y^\lambda) \left[(x+y)^\lambda + x^\lambda + y^\lambda\right] \leq C(xy)^\lambda.$$

Next, for $\lambda \in (0,1]$ and $\phi \in \mathcal{H}_\lambda$, since $|\phi(x)| \leq C(1 + x^\lambda)$ for some constant $C > 0$, we have

$$(x+y)^\lambda |(A\phi)(x,y)| \leq C(x+y)^\lambda \left[3 + (x+y)^\lambda + x^\lambda + y^\lambda\right] \leq C\left(1 + x^{2\lambda} + y^{2\lambda}\right).$$

Finally, for $\lambda \in (0,1]$ and $\phi(x) \in \mathcal{H}^e_\lambda$, there exists $C > 0$ such that $|\phi(x)| \leq Cx^\lambda$ and we have

$$(x \wedge y)^\lambda |(A\phi)(x,y)| \leq C(x \wedge y)^\lambda \left[(x+y)^\lambda + x^\lambda + y^\lambda\right] \leq C(xy)^\lambda.$$

□

Lemma A.2. — *Let $\lambda \in (-\infty, 1] \setminus \{0\}$ and $K \in W^{1,\infty}\left((\varepsilon, 1/\varepsilon)^2\right)$ for every $\varepsilon \in (0,1)$. If K satisfies (2.2), then for all $(x,v,y) \in (0,+\infty)^3$:*

$$\text{(A.2)} \quad \begin{aligned} K(v,y)&\left[\mathbb{1}_{(0,x]}(v+y) - \mathbb{1}_{(0,x]}(v) - \mathbb{1}_{(0,x]}(y)\right] \\ &= K(x-y,y)\mathbb{1}_{(0,x]}(v+y) - K(x,y)\mathbb{1}_{(0,x]}(v) \\ &\quad - \int_v^{+\infty} \partial_x K(z,y)\left[\mathbb{1}_{(0,x]}(z+y) - \mathbb{1}_{(0,x]}(z) - \mathbb{1}_{(0,x]}(y)\right] dz. \end{aligned}$$

If K satisfies (2.3) or (2.4), then for all $(x,v,y) \in (0,+\infty)^3$:

$$\text{(A.3)} \quad \begin{aligned} K(v,y)&\left[\mathbb{1}_{(x,+\infty)}(v+y) - \mathbb{1}_{(x,+\infty)}(v) - \mathbb{1}_{(x,+\infty)}(y)\right] \\ &= K(x-y,y)\mathbb{1}_{x>y}\mathbb{1}_{(x,+\infty)}(v+y) - K(x,y)\mathbb{1}_{(x,+\infty)}(v) \\ &\quad + \int_0^v \partial_x K(z,y)\left[\mathbb{1}_{(x,+\infty)}(z+y) - \mathbb{1}_{(x,+\infty)}(z) - \mathbb{1}_{(x,+\infty)}(y)\right] dz. \end{aligned}$$

Proof. — For $\lambda \in (-\infty, 1] \setminus \{0\}$ we have that $K(\cdot,\cdot)$ and its weak partial derivatives belong to $L^\infty\left((\varepsilon, 1/\varepsilon)^2\right)$, whence, for all $0 < a \leq b < +\infty$ and for all $y > 0$ (see for exemple [**16**]):

$$\text{(A.4)} \quad \int_a^b \partial_x K(z,y) dz = K(b,y) - K(a,y).$$

First assume (2.2), and fix $\lambda \in (-\infty, 0)$. Remark that:

$$\int_a^{+\infty} \partial_x K(z,y) dz = \lim_{b \to +\infty} \int_a^b \partial_x K(z,y) dz = \lim_{b \to +\infty} K(b,y) - K(a,y) = -K(a,y).$$

Hence,

$$\begin{aligned} \int_v^{+\infty} \partial_x K(z,y)\mathbb{1}_{(0,x]}(z+y) dz &= \mathbb{1}_{x>y}\mathbb{1}_{v \leq x-y} \int_0^{+\infty} \partial_x K(z,y)\mathbb{1}_{v \leq z \leq x-y}\, dz \\ &= \mathbb{1}_{(0,x]}(v+y)\left[K(x-y,y) - K(v,y)\right]. \end{aligned}$$

Next,

$$\begin{aligned} -\int_v^{+\infty} \partial_x K(z,y) \mathbb{1}_{(0,x]}(z) dz &= -\mathbb{1}_{v\le x} \int_0^{+\infty} \partial_x K(z,y) \mathbb{1}_{v\le z\le x}\, dz \\ &= \mathbb{1}_{(0,x]}(v)\, [K(v,y) - K(x,y)]\,, \end{aligned}$$

$$\begin{aligned} -\int_v^{+\infty} \partial_x K(z,y) \mathbb{1}_{(0,x]}(y) dz &= -\mathbb{1}_{y\le x} \int_0^{+\infty} \partial_x K(z,y) \mathbb{1}_{v\le z}\, dz \\ &= \mathbb{1}_{(0,x]}(y) K(v,y). \end{aligned}$$

Adding these three terms to the terms on the right-hand of (A.2) the result follows.

Next, assume (2.3) or (2.4). Observe that for $(x,y,z) \in (0,+\infty)^3$, we have

$$\mathbb{1}_{z>x-y} - \mathbb{1}_{y>x} = \mathbb{1}_{y\le x}\mathbb{1}_{z>x-y}. \tag{A.5}$$

Thus,

$$\begin{aligned} &\int_0^v \partial_v K(z,y)\, [\mathbb{1}_{z+y>x} - \mathbb{1}_{z>x} - \mathbb{1}_{y>x}]\, dz \\ &\quad = \int_0^v \partial_x K(z,y)\, [\mathbb{1}_{y\le x}\mathbb{1}_{z>x-y} - \mathbb{1}_{z>x}]\, dz \\ &\quad = \mathbb{1}_{y\le x}\mathbb{1}_{v>x-y} \int_{x-y}^v \partial_x K(z,y) dz - \mathbb{1}_{v>x} \int_x^v \partial_x K(z,y) dz \\ &\quad = \mathbb{1}_{y\le x}\mathbb{1}_{v>x-y}\, [K(v,y) - K(x-y,y)] - \mathbb{1}_{v>x}\, [K(v,y) - K(x,y)] \\ &\quad = [\mathbb{1}_{v>x-y} - \mathbb{1}_{y>x}]\, K(v,y) - \mathbb{1}_{y<x}\mathbb{1}_{v>x-y} K(x-y,y) \\ &\qquad\qquad - \mathbb{1}_{v>x}\, [K(v,y) - K(x,y)]\,. \end{aligned}$$

Adding these terms to the remaining terms on the right-hand of (A.3), the result follows. □

Now we will show a lemma which is useful to show Proposition A.4 stating that the α-moments of μ_0 and μ_0^n remain bounded in time.

Lemma A.3. — *Consider $\alpha \in \mathbb{R}$, $\lambda \in (-\infty, 1]$ and a kernel K satisfying either (2.2), (2.3) or (2.4). We set $\vartheta(x) = x^\alpha$. Then,*

(i) if $\alpha \in (-\infty, 1]$, $(A\vartheta)(x,y) \le 0$, for $(x,y) \in (0,+\infty)^2$,
(ii) if $\alpha \in (1,+\infty)$, $K(x,y)\, |(A\vartheta)(x,y)| \le C_{\lambda,\alpha} \left(x^\alpha y^\lambda + x^\lambda y^\alpha\right)$, for $(x,y) \in (0,+\infty)^2$,

where $C_{\lambda,\alpha}$ is a positive constant depending on λ, α and κ_0.

Proof. — Point *(i)* is obvious, since for $\alpha \le 1$, $(x+y)^\alpha - x^\alpha - y^\alpha \le (x^\alpha + y^\alpha) - x^\alpha - y^\alpha = 0$.

Next, if $\alpha > 1$, using (2.2), (2.3) or (2.4), there holds $K(x,y) \le \kappa_0(x^\lambda + y^\lambda)$. We get

$$\begin{aligned} K(x,y)|(A\vartheta)(x,y)| &\le \kappa_0 x^\lambda\, [|(x+y)^\alpha - x^\alpha| + y^\alpha] + \kappa_0 y^\lambda\, [|(x+y)^\alpha - y^\alpha| + x^\alpha] \\ &\le \alpha\kappa_0 \left[\left(x^\lambda y^\alpha + x^\alpha y^\lambda\right) + (x+y)^{\alpha-1}\left(x^\lambda y + x y^\lambda\right)\right] \\ &\le C \left[\left(x^\lambda y^\alpha + x^\alpha y^\lambda\right) + \left(x^{\alpha-1} + y^{\alpha-1}\right)\left(x^\lambda y + x y^\lambda\right)\right] \\ &\le C \left(x^\lambda y^\alpha + x^\alpha y^\lambda + x^{\lambda+\alpha-1} y + x y^{\lambda+\alpha-1}\right). \end{aligned}$$

Note that $x^{\lambda+\alpha-1} y = x^\alpha y^\lambda \left(\frac{y}{x}\right)^{1-\lambda} = x^\lambda y^\alpha \left(\frac{x}{y}\right)^{\alpha-1} \le x^\alpha y^\lambda \mathbb{1}_{x>y} + x^\lambda y^\alpha \mathbb{1}_{x\le y}$. We have an equivalent bound for the fourth term and the result follows. □

Proposition A.4. — *Consider $\lambda \in (-\infty, 1] \setminus \{0\}$ and a coagulation kernel K satisfying either (2.2), (2.3) or (2.4). Let $\mu_0 \in \mathcal{M}_\lambda^+$, and denote by $(\mu_t)_{t\in[0,T)}$ the (μ_0, K, λ)-weak solution to Smoluchowski's equation. Let μ_0^n be a deterministic discrete measure and $(\mu_t^n)_{t\geq 0}$ the associated (n, K, μ_0^n)-Marcus-Lushnikov process. Let $\alpha \in \mathbb{R}$, then*

(a) if $\alpha \leq 1$, $t \mapsto M_\alpha(\mu_t)$ and $t \mapsto M_\alpha(\mu_t^n)$ are a.s. non-increasing;

(b) if $\alpha > 1$, there exists a positive constant $C_{\lambda,\alpha}$ depending on λ, α and κ_0 such that $M_\alpha(\mu_t) \leq M_\alpha(\mu_0) \exp[t\, C_{\lambda,\alpha} M_\lambda(\mu_0)]$ and $\mathbb{E}[M_\alpha(\mu_t^n)] \leq M_\alpha(\mu_0^n) \exp[t\, C_{\lambda,\alpha} M_\lambda(\mu_0^n)]$.

Proof. — Let $\phi(x) = x^\alpha$. For point *(a)*, first consider (2.8). From Lemma A.3.–*(i)*, we immediately deduce

$$\frac{d}{dt}\langle \mu_t(dx), \phi(x)\rangle = \frac{d}{dt} M_\alpha(\mu_t) = \frac{1}{2}\langle \mu_t(dx)\mu_t(dy), (A\phi)(x,y)K(x,y)\rangle \leq 0.$$

Next, consider (2.10) and remark that $\phi\left(X_{s-}^i + X_{s-}^j\right) - \phi\left(X_{s-}^i\right) - \phi\left(X_{s-}^j\right) = (A\phi)\left(X_{s-}^i, X_{s-}^j\right)$. From Lemma A.3.–*(i)* and since J is a positive measure, we deduce that the jumps of $M_\alpha(\mu_t^n) = \langle \mu_t^n(dx), \phi(x)\rangle$ are negative and the conclusion follows.

For point *(b)*, consider (2.8). According to Lemma A.3.–*(ii)*, we deduce:

$$\begin{aligned}
\frac{d}{dt} M_\alpha(\mu_t) = \frac{1}{2}\langle \mu_t(dx)\mu_t(dy), (A\phi)(x,y)K(x,y)\rangle &\leq \frac{C_{\lambda,\alpha}}{2}\left\langle \mu_t(dx)\mu_t(dy), x^\alpha y^\lambda + x^\lambda y^\alpha\right\rangle \\
&\leq C_{\lambda,\alpha} M_\lambda(\mu_t) M_\alpha(\mu_t) \\
&\leq C_{\lambda,\alpha} M_\lambda(\mu_0) M_\alpha(\mu_t),
\end{aligned}$$

we used the point *(a)*. We conclude using the Gronwall lemma.

Next, we take the expectation in (2.11). Remarking that $(A\phi)(x,x) \geq 0$, using Lemma A.3.–*(ii)*, since μ_0^n is deterministic, and since $M_\alpha(\mu_t^n) = \langle \mu_t^n(dx), \phi(x)\rangle$, we deduce:

$$\begin{aligned}
\mathbb{E}[M_\alpha(\mu_t^n)] &= M_\alpha(\mu_0^n) + \frac{1}{2}\int_0^t \mathbb{E}[\langle \mu_s^n(dx)\mu_s^n(dy), (A\phi)(x,y)K(x,y)\rangle]\, ds \\
&\qquad - \frac{1}{2n}\int_0^t \mathbb{E}[\langle \mu_s^n(dx), (A\phi)(x,x)K(x,x)\rangle]\, ds, \\
&\leq M_\alpha(\mu_0^n) + \frac{C_{\lambda,\alpha}}{2}\int_0^t \mathbb{E}\left[\left\langle \mu_t^n(dx)\mu_t^n(dy), x^\alpha y^\lambda + x^\lambda y^\alpha\right\rangle\right] ds \\
&\leq M_\alpha(\mu_0^n) + C_{\lambda,\alpha}\int_0^t \mathbb{E}[M_\lambda(\mu_s^n) M_\alpha(\mu_s^n)]\, ds \\
&\leq M_\alpha(\mu_0^n) + C_{\lambda,\alpha} M_\lambda(\mu_0^n)\int_0^t \mathbb{E}[M_\alpha(\mu_s^n)]\, ds,
\end{aligned}$$

where we used the point *(a)*. We conclude using the Gronwall lemma. □

Finally, we present the

Proof of Lemma 5.1. — Assume that $\lambda \in (0,1]$ and recall (5.1). First, for *(i)* and *(ii)*, by direct integration, we have

$$\begin{aligned}
\int_0^{+\infty} x^{\lambda-1}\theta_{(x)}^n dx &= \frac{1}{\sqrt{n}}\int_0^1 x^{\lambda-1} dx + \frac{1}{\sqrt{n}}\int_1^{+\infty} x^{-\lambda-\varepsilon-1} dx \\
&= \frac{1}{\lambda\sqrt{n}} + \frac{1}{(\lambda+\varepsilon)\sqrt{n}} \leq \frac{2}{\lambda\sqrt{n}},
\end{aligned}$$

and

$$\int_0^{+\infty} x^{2\lambda-1}\theta^n_{(x)}dx = \frac{1}{\sqrt{n}}\int_0^1 x^{2\lambda-1}dx + \frac{1}{\sqrt{n}}\int_1^{+\infty} x^{-\varepsilon-1}dx$$
$$= \frac{1}{2\lambda\sqrt{n}} + \frac{1}{\varepsilon\sqrt{n}} \leq \frac{(\lambda+\varepsilon)}{\lambda\varepsilon\sqrt{n}}.$$

Next, for *(iii)* we have:

$$A_n(v,y) = v^\lambda \int_0^{+\infty} \frac{x^{\lambda-1}}{\theta^n_{(x)}}\left(\mathbb{1}_{x<v\wedge y} + \mathbb{1}_{v\vee y<x<v+y}\right)dx = v^\lambda \int_0^{v\wedge y} \frac{x^{\lambda-1}}{\theta^n_{(x)}}dx + v^\lambda \int_{v\vee y}^{v+y} \frac{x^{\lambda-1}}{\theta^n_{(x)}}dx$$
$$:= I_n(v,y) + J_n(v,y).$$

We have the following bounds: if $v\wedge y \leq 1$, then

$$I_n(v,y) = v^\lambda \int_0^{v\wedge y} \sqrt{n}\, x^{\lambda-1}dx = \frac{\sqrt{n}}{\lambda}v^\lambda (v\wedge y)^\lambda \leq \frac{\sqrt{n}}{\lambda}v^\lambda y^\lambda.$$

Next, if $v\wedge y > 1$,

$$I_n(v,y) = v^\lambda \int_0^1 \sqrt{n}\, x^{\lambda-1}dx + v^\lambda \int_1^{v\wedge y} \sqrt{n}\, x^{3\lambda+\varepsilon-1}dx$$
$$\leq \sqrt{n}\, v^\lambda \left[\frac{1}{\lambda} + \frac{(v\wedge y)^{3\lambda+\varepsilon}}{3\lambda+\varepsilon}\right]$$
$$\leq \frac{\sqrt{n}}{\lambda}\left[v^\lambda + (v\wedge y)^{3\lambda+\varepsilon}v^\lambda\right]$$
$$\leq \frac{\sqrt{n}}{\lambda}\left[v^\lambda y^\lambda + (v\wedge y)^{2\lambda}(v\vee y)^{2\lambda+\varepsilon}\mathbb{1}_{\lambda\in(0,1/2)} + (v\wedge y)(v\vee y)^{4\lambda+\varepsilon-1}\mathbb{1}_{\lambda\in[1/2,1]}\right].$$

Thus, in any case

$$I_n(v,y) = \frac{\sqrt{n}}{\lambda}\left[v^\lambda y^\lambda + (v\wedge y)^{2\lambda}(v\vee y)^{2\lambda+\varepsilon}\mathbb{1}_{\lambda\in(0,1/2)} + (v\wedge y)(v\vee y)^{4\lambda+\varepsilon-1}\mathbb{1}_{\lambda\in[1/2,1]}\right].$$

Next, since $x^{\lambda-1}$ and $\theta^n_{(x)}$ are non-increasing functions, according to the mean value theorem, we deduce that $J_n(v,y) \leq v^\lambda \left(\frac{(v\vee y)^{\lambda-1}}{\theta^n_{(v+y)}}\right)(v\wedge y)$.

First, assume that $v+y<1$, then we get $J_n(v,y) \leq \sqrt{n}\, v^\lambda (v\vee y)^{\lambda-1}(v\wedge y) \leq \sqrt{n}\, v^\lambda y^\lambda$.

Next, assume that $v+y \geq 1$, then

$$J_n(v,y) \leq \sqrt{n}\, v^\lambda (v\vee y)^{\lambda-1}(v+y)^{2\lambda+\varepsilon}(v\wedge y) \leq 2^{2\lambda+\varepsilon}\sqrt{n}\, v^\lambda (v\wedge y)(v\vee y)^{3\lambda+\varepsilon-1}$$
$$\leq 2^{2\lambda+\varepsilon}\sqrt{n}\left[(v\wedge y)^{2\lambda}(v\vee y)^{2\lambda+\varepsilon}\mathbb{1}_{\lambda\in(0,1/2)} + (v\wedge y)(v\vee y)^{4\lambda+\varepsilon-1}\mathbb{1}_{\lambda\in[1/2,1]}\right].$$

When $\lambda\in(0,1/2)$, we used $(v\wedge y) \leq (v\wedge y)^{2\lambda}(v\vee y)^{1-2\lambda}$ to deduce the bound $v^\lambda(v\wedge y)(v\vee y)^{3\lambda+\varepsilon-1} \leq v^\lambda (v\wedge y)^{2\lambda}(v\vee y)^{1-2\lambda}(v\vee y)^{3\lambda+\varepsilon-1} \leq (v\wedge y)^{2\lambda}(v\vee y)^{2\lambda+\varepsilon}$.

Thus, in any case

$$J_n(v,y) \leq \sqrt{n}v^\lambda y^\lambda + 2^{2\lambda+\varepsilon}\sqrt{n}\left[(v\wedge y)^{2\lambda}(v\vee y)^{2\lambda+\varepsilon}\mathbb{1}_{\lambda\in(0,1/2)} + (v\wedge y)(v\vee y)^{4\lambda+\varepsilon-1}\mathbb{1}_{\lambda\in[1/2,1]}\right].$$

Finally, we deduce the bound:

$$A_n(v,y) \leq \frac{2\sqrt{n}}{\lambda} v^\lambda y^\lambda$$
$$+\sqrt{n}\left(2^{2\lambda+\varepsilon}+\frac{1}{\lambda}\right)\left[(v\wedge y)^{2\lambda}(v\vee y)^{2\lambda+\varepsilon}\mathbb{1}_{\lambda\in(0,1/2)}+(v\wedge y)(v\vee y)^{4\lambda+\varepsilon-1}\mathbb{1}_{\lambda\in[1/2,1]}\right].$$

This concludes the proof of Lemma 5.1. □

References

[1] D.J. Aldous. Deterministic and Stochastic Models for Coalescence (Aggregation, Coagulation): A Review of the Mean-Field Theory of Probabilists. *Bernoulli*, 5:3–48, 1999.

[2] M. Deaconu, N. Fournier, and E. Tanré. Rate of Convergence of a Stochastic Particle System for the Smoluchowski Coagulation equation. *Methodology and Computing in Applied Probability*, (5):131–158, 2003.

[3] A. Eibeck and W. Wagner. Stochastic Particle Approximations for Smoluchowski's Coagulation Equation. *Markov Processes and Related Fields*, (9):103–130, 2003.

[4] N. Fournier. A distance for coagulation. *Stoch. Proc. Appl.*, 12(2):399–406, 2006.

[5] N. Fournier. On some stochastic coalescents. *Proba. Theory Related Fields*, 136(4):509–523, 2006.

[6] N. Fournier and J. S. Giet. Convergence of the Marcus-Lushnikov process. *Methodology and Computing in Applied Probability*, 6:219–231, 2004.

[7] N. Fournier and Ph. Laurençot. Well-possedness of Smoluchowski's Coagulation Equation for a Class of Homogeneous Kernels. *J. Functional Analysis*, 233(2):351–379, 2006.

[8] N. Fournier and E. Löcherbach. Stochastic coalescence with homogeneous-like interaction rates. *Stoch. Proc. Appl.*, 119:45–73, 2009.

[9] J. Jacod and A. N. Shiryaev. *Limit theorems for stochastics processes*. Springer-Verlag, 2003.

[10] I. Jeon. Existence of Gelling Solutions for Coagulation-Fragmentation Equations. *Comm. Math. Phys.*, 3:541–567, 1998.

[11] B. Jourdain. Nonlinear Process Associated with the Discrete Smoluchowski Coagulation-Fragmentation Equation. *Markov Processes and Related Fields*, (9):103–130, 2003.

[12] V. Kolokoltsov. The Central Limit Theorem for the Smoluchowski Coagulation Model. *arXiv:0708.0329v1 [math.PR]. Probability Theory and Related Fields*, (1):87–153, 2010.

[13] A. Lushnikov. Some New Aspects of Coagulation Theory. *Izv. Akad. Nauk SSSR, Ser. Fiz. Atmosfer. I Okeana*, 14:738–743, 1978.

[14] A. Marcus. Stochastic Coalescence. *Technometrics*, 10:78–109, 1968.

[15] J.R. Norris. Smoluchowski's coagulation equation: uniqueness, non-uniqueness and hydrodynamic limit for the stochastic coalescent. *Ann. Appl. Probab.*, 9(1):78–109, 1999.

[16] W. P. Ziemer. *Weakly differentiable functions, Graduate Texts in Mathematics*. Springer-Verlag, 1989.

Eduardo Cepeda and Nicolas Fournier, Laboratoire d'Analyse et de Mathématiques Appliquées, UMR 8050. Université Paris-Est. 61, avenue du Général de Gaulle, 94010 Créteil Cédex
E-mail : `eduardo.cepeda-chiluisa@etu.univ-paris12.fr`, `nicolas.fournier@univ-paris12.fr`