# WELL-POSEDNESS FOR A COAGULATION MULTIPLE-FRAGMENTATION EQUATION

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Abstract. We consider a coagulation multiple-fragmentation equation, which describes the concentration $c_t(x)$ of particles of mass $x \in (0, \infty)$ at the instant $t \geq 0$ in a model where fragmentation and coalescence phenomena occur. We study the existence and uniqueness of measured-valued solutions to this equation for homogeneous-like kernels of homogeneity parameter $\lambda \in (0, 1]$ and bounded fragmentation kernels, although a possibly infinite total fragmentation rate, in particular an infinite number of fragments, is considered. This work relies on the use of a Wasserstein-type distance, which has shown to be particularly well-adapted to coalescence phenomena. It was introduced in previous works on coagulation and coalescence.

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## 1. Introduction

The coagulation-fragmentation equation is a deterministic equation that models the evolution in time of a system of a very big number of particles (mean-field description) undergoing coalescences and fragmentations. The particles in the system grow and decrease due to successive mergers and dislocations, each particle is fully identified by its mass $x \in (0, \infty)$, we do not consider its position in space, its shape nor other geometrical properties. Examples of applications of these models arise in polymers, aerosols and astronomy.

In these notes we are interested in the phenomena of coagulation and fragmentation at microscopic scale, we will describe the evolution of the concentration of particles of mass x in the following way. On the one hand, the coalescence of two particles of mass $x$ and $y$ gives birth a new one of mass $x + y$, $\{x, y\} \to x + y$ with a rate proportional to the coagulation kernel $K(x, y)$. On the other hand, the fragmentation of a particle of mass $x$ gives birth a new set of smaller particles $x \to \{\theta_1 x, \theta_2 x, \ldots\}$, where $\theta_i x$ represents the fragments of $x$, with a rate proportional to $F(x)\beta(d\theta)$ and where $F : (0, \infty) \to (0, \infty)$ and $\beta$ is a positive measure on the set $\Theta = \{\theta = (\theta_i)_{i\geq 1} : 1 > \theta_1 \geq \theta_2 \geq \ldots \geq 0\}$. This means that the distribution of the ratios of daughter masses to parent mass is only determined by a function of these ratios (and not by the parent mass). Denoting $c_t(x)$ the concentration of particles of mass $x \in (0, \infty)$ at time $t$, the dynamics of $c$ is given by

$$\partial_t c_t(x) = \frac{1}{2}\int_0^x K(y, x-y)c_t(y)c_t(x-y)\,dy - c_t(x)\int_0^\infty K(x, y)c_t(y)\,dy + \int_\Theta \left[\sum_{i=1}^\infty \frac{1}{\theta_i} F\left(\frac{x}{\theta_i}\right) c_t\left(\frac{x}{\theta_i}\right) - F(x)c_t(x)\right]\beta(d\theta). \tag{1.1}$$

The fragmentation part of the model was first introduced by Bertoin [6] and takes into account an infinite measure $\beta$ and a mechanism of dislocation with a possibly infinite number of fragments.

The macroscopic scale version of this model (wich is intrinsecally stochastic) is studied in Cepeda [8]. We believe that a *hydrodynamical limit* result concerning this two settings is possible to obtain in the following way. Denoting by $\mu^n = \frac{1}{n}\sum_{i\geq 1}\delta_{m_i}$ the empirical measure associated to the system composed by $(m_1, m_2, \ldots)$, then the Coalescence-Fragmentation process associated $(\mu^n_t)_{t\geq 0}$ converges to the solution to equation (1.1). For a first result concerning convergence in the case where $F \equiv 0$ see Norris [23, 24] and Cepeda-Fournier [9] for a explicit rate of convergence. Nevertheless, this is not the aim of these notes.

In this paper we are mainly interested in a result of general well-posedness, this means, with the less possible assumptions on $K$, $F$, $\beta$ and the initial condition. Our method is based on the use of the two following distance: for $\lambda \in (0,1]$ and $c, d$ two positive Radon measures such that $\int_0^\infty x^\lambda (c+d)(dx) < \infty$, we set

$$d_\lambda(c,d) = \int_0^\infty x^{\lambda-1}\left|c((x,\infty)) - d((x,\infty))\right| dx.$$

In this paper we extend the result in Fournier-Laurençot [15] concerning only coagulation, and we show existence and uniqueness to (1.1) for a class of homogeneous-like coagulation kernels and bounded fragmentation kernels, in the class of measures having a finite moment of order the degree of homogeneity of the coagulation kernel. Unfortunately this method does not extend to unbounded fragmentation kernels. Our assumptions on $F$ are not very restrictive for small masses, since we do not ask to $F$ to be zero on a neighbourhood of 0. On the other hand, we control the big masses imposing to the fragmentation kernel to be bounded near infinity. Nevertheless, we are able to consider infinite total fragmentation rates for all $x > 0$.

We have chosen this model for the fragmentation since it is actually more tractable mathematically, see Bertoin [6, 5] and Haas [18, 19] where the properties of the only fragmentation model are extensively studied. Kolokoltsov [20] shows in the discrete case a hydrodynamical limit result for a different model than ours, namely he introduces a mass exchange Markov process. An extensive study of the methods used by the author are given in the books [22, 21]. Finally, we refer to Eibeck-Wagner [12] where a different model is studied which is used to approach general nonlinear kinetic equations.

The paper is organized as follows: we introduce some notation, definitions and the result in Sections 2 and the proof is given in Section 4, we compare our result to those known to us in Section 3.

## 2. The Coagulation multi-Fragmentation equation.- Notation, Definitions and Result

We first give some notation and definitions. We consider the set of non-negative Radon measures $\mathcal{M}^+$ and for $\lambda \in \mathbb{R}$ and $c \in \mathcal{M}^+$, we set

$$M_\lambda(c) := \int_0^\infty x^\lambda c(dx), \qquad \mathcal{M}^+_\lambda = \left\{c \in \mathcal{M}^+,\ M_\lambda(c) < \infty\right\}. \tag{2.1}$$

Next, for $\lambda \in (0,1]$ we introduce the space $\mathcal{H}_\lambda$ of test functions,

$$\mathcal{H}_\lambda = \left\{ \phi \in \mathcal{C}([0,\infty)) \text{ such that } \phi(0)=0 \text{ and } \sup_{x\neq y} \frac{|\phi(x)-\phi(y)|}{|x-y|^\lambda} < \infty \right\}.$$

Note that $\mathcal{C}^1_c((0,\infty)) \subset H_\lambda$.

Here and below, we use the notation $x \wedge y := \min\{x,y\}$ and $x \vee y := \max\{x,y\}$ for $(x,y) \in (0,\infty)^2$.

**Hypothesis 2.1** (Coagulation and Fragmentation Kernels)**.** *Consider $\lambda \in (0,1]$ and a symmetric coagulation kernel $K : (0,\infty)\times(0,\infty) \to [0,\infty)$ i.e., $K(x,y) = K(y,x)$. Assume that $K$ is locally Lipschitz, more precisely assume that it belongs to $W^{1,\infty}((\varepsilon,1/\varepsilon)^2)$ for every $\varepsilon > 0$ and that it satisfies*

$$K(x,y) \leq \kappa_0 (x+y)^\lambda, \tag{2.2}$$

$$(x^\lambda \wedge y^\lambda)|\partial_x K(x,y)| \leq \kappa_1 x^{\lambda-1} y^\lambda, \tag{2.3}$$

*for all $(x,y) \in (0,\infty)^2$ and for some positive constants $\kappa_0$ and $\kappa_1$. Consider also a fragmentation kernel $F : (0,\infty) \to [0,\infty)$ and assume that $F$ belongs to $W^{1,\infty}((\varepsilon,1/\varepsilon))$ for every $\varepsilon > 0$ and that it satisfies*

$$F(x) \leq \kappa_2, \tag{2.4}$$

$$|F'(x)| \leq \kappa_3\, x^{-1}, \tag{2.5}$$

*for $x \in (0,\infty)$ and some positive constants $\kappa_2$ and $\kappa_3$.*

For example, the coagulation kernels listed below, taken from the mathematical and physical literature, satisfy Hypothesis 2.1.

$$\begin{array}{lll} K(x,y) = (x^\alpha + y^\alpha)^\beta & \text{with } \alpha \in (0,\infty),\ \beta \in (0,\infty) \text{ and } \lambda = \alpha\beta \in (0,1], \\ K(x,y) = x^\alpha y^\beta + x^\beta y^\alpha & \text{with } 0 \leq \alpha \leq \beta \leq 1 \text{ and } \lambda = \alpha+\beta \in (0,1], \\ K(x,y) = (xy)^{\alpha/2}(x+y)^{-\beta} & \text{with } \alpha \in (0,1],\ \beta \in [0,\infty) \text{ and } \lambda = \alpha-\beta \in (0,1], \\ K(x,y) = (x^\alpha + y^\alpha)^\beta |x^\gamma - y^\gamma| & \text{with } \alpha \in (0,\infty),\ \beta \in (0,\infty),\ \gamma \in (0,1] \text{ and } \lambda = \alpha\beta+\gamma \in (0,1], \\ K(x,y) = (x+y)^\lambda e^{-\beta(x+y)^{-\alpha}} & \text{with } \alpha \in (0,\infty),\ \beta \in (0,\infty), \text{ and } \lambda \in (0,1]. \end{array}$$

On the other hand, the following fragmentation kernels satisfy Hypothesis 2.1.

$F(x) \equiv 1$,
all non-negative function $F \in C^2(0,\infty)$, bounded, convex and non-increasing,
all non-negative function $F \in C^2(0,\infty)$, bounded, concave and non-decreasing.

We define the set of ratios by

$$\Theta = \{\theta = (\theta_k)_{k\geq 1} : 1 > \theta_1 \geq \theta_2 \geq \ldots \geq 0\}.$$

**Hypothesis 2.2** (The $\beta$ measure.-)**.** *We consider on $\Theta$ a measure $\beta(\cdot)$ and assume that it satisfies*

$$\beta\left(\sum_{k\geq 1} \theta_k > 1\right) = 0, \tag{2.6}$$

$$C^\lambda_\beta := \int_\Theta \left[\sum_{k\geq 2} \theta_k^\lambda + (1-\theta_1)^\lambda\right] \beta(d\theta) < \infty, \quad \textit{for some } \lambda \in (0,1]. \tag{2.7}$$

**Remark 2.3.** *i) The property (2.6) means that there is no gain of mass due to the dislocation of a particle. Nevertheless, it does not exclude a loss of mass due to the dislocation of the particles.*

*ii) Note that under (2.6) we have* $\sum_{k\geq 1}\theta_k - 1 \leq 0$ *$\beta$-a.e., and since* $\theta_k \in [0,1)$ *for all* $k \geq 1$, $\theta_k \leq \theta_k^\lambda$, *we have*

$$\begin{cases} 1-\theta_1^\lambda \leq 1-\theta_1 \leq (1-\theta_1)^\lambda, \ \beta - a.e., \\ \\ \sum_{k\geq 1}\theta_k^\lambda - 1 = \sum_{k\geq 2}\theta_k^\lambda - (1-\theta_1^\lambda) \leq \sum_{k\geq 2}\theta_k^\lambda, \ \beta - a.e. \end{cases} \tag{2.8}$$

*implying the following bounds:*

$$\begin{cases} \displaystyle\int_\Theta (1-\theta_1)\beta(d\theta) \leq C_\beta^\lambda, \ \int_\Theta \left[\sum_{k\geq 2}\theta_k^\lambda + (1-\theta_1^\lambda)\right]\beta(d\theta) \leq C_\beta^\lambda, \\ \displaystyle\int_\Theta \left(\sum_{k\geq 1}\theta_k^\lambda - 1\right)^+ \beta(d\theta) \leq C_\beta^\lambda. \end{cases} \tag{2.9}$$

*We point out that* $\int_\Theta \left|\sum_{k\geq 1}\theta_k^\lambda - 1\right| \beta(d\theta) \leq 2C_\beta^\lambda$ *but when the term* $\sum_{k\geq 1}\theta_k^\lambda - 1$ *is negative our calculations can be realized in a simpler way. We will thus use the positive bound given in the last inequality.*

The result of the deterministic framework depends strongly on the use of the distance which is defined for $\lambda \in (0,1]$ and $c$, $d \in \mathcal{M}_\lambda^+$ (recall 2.1) the distance

$$d_\lambda(c,d) = \int_0^\infty x^{\lambda-1}\left|\int_x^\infty (c(dy)-d(dy))\right| dx. \tag{2.10}$$

**Definition 2.4** (Weak solution to (1.1))**.** *Let* $c^{in} \in \mathcal{M}_\lambda^+$. *A family* $(c_t)_{t\geq 0} \subset \mathcal{M}^+$ *is a* $(c^{in}, K, F, \beta, \lambda)$*-weak solution to (1.1) if* $c_0 = c^{in}$,

$$t \mapsto \int_0^\infty \phi(x)c_t(dx) \quad \textit{is differentiable on } [0,\infty)$$

*for each* $\phi \in \mathcal{H}_\lambda$, *and for every* $t \in [0,\infty)$,

$$\sup_{s\in[0,t]} M_\lambda(c_s) < \infty, \tag{2.11}$$

*and for all* $\phi \in \mathcal{H}_\lambda$

$$\begin{aligned} \frac{d}{dt}\int_0^\infty \phi(x)c_t(dx) &= \frac{1}{2}\int_0^\infty\int_0^\infty K(x,y)(A\phi)(x,y)c_t(dx)c_t(dy) \\ &\quad + \int_0^\infty F(x)\int_\Theta (B\phi)(\theta,x)\beta(d\theta)c_t(dx), \end{aligned} \tag{2.12}$$

*where the functions* $(A\phi) : (0,\infty)\times(0,\infty) \to \mathbb{R}$ *and* $(B\phi) : \Theta \times (0,\infty) \to \mathbb{R}$ *are defined by*

$$(A\phi)(x,y) = \phi(x+y) - \phi(x) - \phi(y), \tag{2.13}$$

$$(B\phi)(\theta,x) = \sum_{i=1}^\infty \phi(\theta_i x) - \phi(x). \tag{2.14}$$

This equation can be split into two parts, the first integral explains the evolution in time of the system under coagulation and the second integral explains the behaviour of the system when undergoing fragmentation and it corresponds to a growth in the number of particles of masses $\theta_1 x$, $\theta_2 x$, ..., and to a decrease in the number of particles of mass $x$ as a consequence of their fragmentation.

According to (2.2), (2.4), Lemma 4.1. below, (2.11) and (2.7), the integrals in (2.12) are absolutely convergent and bounded with respect to $t \in [0, s]$ for every $s \geq 0$.

The main result reads as follows.

**Theorem 2.5.** *Consider $\lambda \in (0, 1]$ and $c^{in} \in \mathcal{M}_\lambda^+$. Assume that the coagulation kernel $K$, the fragmentation kernel $F$ and the measure $\beta$ satisfy Hypotheses 2.1. and 2.2 with the same $\lambda$.*

*Then, there exists a unique $(c^{in}, K, F, \beta, \lambda)$-weak solution to (1.1).*

It is important to note that the main interest of this result is that only one moment is asked to the initial condition $c^{in}$. The assumptions on the coagulation kernel $K$ and the measure $\beta$ are reasonable. Whereas the main limitation is that we need to assume that the fragmentation kernel is bounded. It is also worth to point out that we have chosen to study this version of the equation because of its easy physical intuition.

## 3. Other formulations for the fragmentation equation

To enable us to compare our results to those obtained in other works, we discuss the relationships between the various formulations. The first works (see [1, 10, 14]) were concentrated on the binary fragmentation where the particles dislocate only into two particles:

**Binary Model .-** Denoting $c_t(x)$ the concentration of particles of mass $x \in (0, \infty)$ at time $t$, the dynamics of the fragmentation is given by the operator

$$(\mathcal{F}_b c_t)(x) \quad = \quad \int_x^\infty F_b(x, y-x) c_t(y) dy - \frac{1}{2} c_t(x) \int_0^x F_b(y, x-y) dy,$$

for $(t, x) \in (0, \infty)^2$. The binary fragmentation kernel $F_b$ is also a symmetric function and $F_b(x, y)$ is the rate of fragmentation of particles of mass $x + y$ into particles of masses $x$ and $y$.

Note that we can obtain the continuous coagulation binary-fragmentation equation, for example, by considering $\beta$ with support in $\{\theta : \theta_1 + \theta_2 = 1\}$ and $\beta(d\theta) = h(\theta_1)\, d\theta_1 \delta_{\{\theta_2 = 1-\theta_1\}}$, and setting $F_b(x, y) = \frac{2}{x+y} F(x+y) h\left(\frac{x}{x+y}\right)$ where $h(\cdot)$ is a continuous function on $[0, 1]$ and symmetric at $1/2$. Under this framework, one can find some results of existence and uniqueness for example in [11, 25, 26].

**Multifragmentation Model .-** We can consider a version of the coagulation - multi fragmentation equation where the fragmentation operator has the following representation; see [10]:

$$(\mathcal{F}_m c_t)(x) = \int_x^\infty F_m(y, x) c_t(y) dy - c_t(x) \int_0^x \frac{y}{x} F_m(x, y) dy,$$

where $F_m(x, y)$ is the fragmentation kernel and explains the dislocation of a particle $x$ into smaller particles $y$ and $x - y$. In the same spirit in [2, 3, 4, 16, 17] is considered an equivalent representation of the fragmentation operator

$$(\mathcal{F}_m c_t)(x) = \int_0^x b(x, y) a(y) c_t(y) dy - a(x) c_t(x),$$

where $a(x) = \int_0^x \frac{y}{x} F_m(x,y)dy$ is the total rate of fragmentation of a particle of mass $x$, and $b(x,y) = F_m(y,x)/a(y)$ is a non-negative function and represents the distribution (probability) of particles of mass $x$ generated from particles of mass $y \geq x$. This operator allows to consider a multi-fragmentation model in the following way, for each fragmentation of a particle of mass $y$, the average number and mass of the fragments $x$ are, respectively

$$N(y) = \int_0^y b(x,y)dx, \qquad \text{and} \qquad m(y) = \int_0^y xb(x,y)dy, \tag{3.1}$$

and it is usually assumed that no mass is lost when a particle breaks up, that is, $\int_0^y xb(x,y)dy = y$. In both the physics and mathematics literature, concerning the fragmentation operator, particular attention has been paid to models with the following self-similar dynamic:

- $S(x) = Cx^\alpha$, for some constant $C > 0$ and $\alpha \in \mathbb{R}$.
- $b(x,y) = \dfrac{1}{x} h\left(\dfrac{y}{x}\right)$ with $\int_0^1 xh(x)dx = 1$.

The main two reasons for this are that self-similar assumptions are relevant for applications and that they are also more mathematically tractable. There is also a significant literature on probabilistic models for the microscopic mechanism of fragmentation with a self-similar dynamic. We refer to the book by Bertoin [7] for an overview and to the [13, 18, 19] for discussions of the relations between the probabilistic models and the above operator.

Remark that we express the rate of fragmentation of a particle of mass $x$ as the product $F(x)\beta(d\theta)$. If we consider fragmentation kernels of the form $F_m(x,y) = F(x)\frac{1}{x}h\left(\frac{y}{x}\right)$, note that the rate of fragmentation of a particle of mass $x$ is $a(x) = F(x)\int_0^1 h(\theta)d\theta$ which, under our assumptions, can be infinite for all $x$, and denoting $\theta$ the fragments (3.1) becomes

$$(\mathcal{F}_m c_t)(x) = \int_0^1 \left[\frac{1}{\theta^2} F\left(\frac{x}{\theta}\right) c_t\left(\frac{x}{\theta}\right) - F(x)c_t(x)\right] h(\theta)d\theta.$$

Nevertheless, it is not clear the existence of a measure $h$ such that allow the identification

$$(\mathcal{F}_m c_t)(x) = \int_\Theta \left[\sum_{i=1}^\infty \frac{1}{\theta_i} F\left(\frac{x}{\theta_i}\right) c_t\left(\frac{x}{\theta_i}\right) - F(x)c_t(x)\right] \beta(d\theta),$$

which demands some properties to the measure $h$.

On the one hand, one of the difficulties when working with the coagulation-fragmentation equation, as stated in Banasiak-Lamb [3], is that the coagulation operator is not linear. The authors used a compactness method, the method used constrains the authors (see [10, 16, 17]) to require some finite moments to the initial conditions, existence holds in the functional set $X = \{f \in L^1(0,\infty) : \int_0^\infty (1+x)|f(x)|dx < \infty\}$ (and the solutions are not measures), in [4] is required higher moments to treat different fragmentation rates than those found in the other works. It is also needed to control the number of fragments at each dislocation and $\beta$ must be integrable.

It is worth to point out that the method we use in this paper relies on a previous result on the coagulation-only equation, which considers a particular well-adapted distance that allows to relax the hypotheses on the initial condition. The coagulation-only ($F \equiv 0$) equation is known as Smoluchowski's equation and it has been studied by several authors, Norris in [23] gives the first general well-posedness result and Fournier and Laurençot [15] give a result of existence and uniqueness of a measured-valued solution for a class of homogeneous-like kernels. The fragmentation-only ($K \equiv 0$) equation has been studied in Bertoin [6] and Haas [18]. In particular, in [6] the self-similar fragmentations are characterized using a fragmentation kernel of the type $F(x) = x^{\alpha}$ for $\alpha \in \mathbb{R}$ and where the particles may undergo multi-fragmentations.

The main aim of this paper is to extend this result to the case where fragmentation is added to the process. We remark that the model is different and allows us to consider other features of the fragmentation that previous models do not present. Namely, we allow the fragmentation to give an infinity of fragments at each dislocation and the measure $\beta$ is not necessarily integrable. In this sense, although we consider bounded fragmentation kernels, the total fragmentation rate can be infinite for each $x \geq 0$.

Roughly, in [10], an existence and uniqueness result is given for $K(x,y) = r(x)r(y) + \alpha(x,y)$, where $\alpha \in \mathcal{C}([0,\infty)\times[0,\infty))$ is the dominant term for the coagulation-fragmentation process since the kernel $F_m \in \mathcal{C}([0,\infty)\times[0,\infty))$ is assumed to satisfy $F_m(x,y) \leq C(1+\max(x,r(x)))$, for $x,y \geq 0$ and $\int_0^x F_m(x,y) \leq \gamma(x)\max(x,r(x))$, for $x \geq 0$ and $\gamma : [0,\infty) \to [0,\infty)$ with $\gamma(x) \underset{x\to\infty}{\longrightarrow} 0$.

In [16, 17] the coagulation kernel is assumed to satisfy $K(x,y) \leq C(1+x)^{\mu}(1+y)^{\mu}$ with $\mu \in [0,1)$, and $a(x) \leq C_1(1+x)^{a_1}$ and $\int_0^x (1+y)^{1+\nu} b(y,x)dy \leq C_2(1+x)^{a_2}$, where $C_1$ and $C_2$ are positive constants and where $a_1 + a_2 \leq 1+\nu$ with $1+\nu \in (0,1)$.

Finally, in [4] the authors consider $a(x) \leq C_1(1+x^{\mu})$ and $\int_0^x y b(y,x)dx \leq C_2(1+x^{\nu})$, with $\mu,\nu \in [0,\infty)$ and where $C_1$ and $C_2$ are positive constants. This result allows to consider stronger fragmentation rates requiring a stronger moment for the initial condition.

## 4. Proofs

We begin giving some properties of the operators $(A\phi)$ and $(B\phi)$ for $\phi \in \mathcal{H}_\lambda$ which allow us to justify the weak formulation (2.12).

**Lemma 4.1.** *Consider $\lambda \in (0,1]$, $\phi \in \mathcal{H}_\lambda$. Then there exists $C_\phi$ depending on $\phi$, $\theta$ and $\lambda$ such that*

$$
\begin{aligned}
(x+y)^{\lambda}|(A\phi)(x,y)| &\leq C_\phi (xy)^{\lambda},\\
|(B\phi)(\theta,x)| &\leq C_\phi x^{\lambda}\left[\sum_{i\geq 2}\theta_i^{\lambda} + (1-\theta_1)^{\lambda}\right],
\end{aligned}
$$

*for all $(x,y) \in (0,\infty)^2$ and for all $\theta \in \Theta$.*

*Prof of Lemma 4.1.* For $(A\phi)$ we recall [15, Lemma 3.1]. Next, consider $\lambda \in (0,1]$ and $\phi \in \mathcal{H}_\lambda$ then, since $\phi(0) = 0$,

$$
\begin{aligned}
|(B\phi)(\theta,x)| &\leq |\phi(\theta_1 x) - \phi(x)| + \sum_{i\geq 2}|\phi(\theta_i x) - \phi(0)|\\
&\leq C_\phi x^{\lambda}(1-\theta_1)^{\lambda} + C_\phi x^{\lambda}\sum_{i\geq 2}\theta_i^{\lambda}.
\end{aligned}
$$

□

We are going to work with a distance between solutions depending on $\lambda$. The distance $d_\lambda$ (2.10) involves the primitives of the solution of (1.1), thus we recall [15, Lemma 3.2].

**Lemma 4.2.** *For $c \in \mathcal{M}^+$ and $x \in (0,\infty)$, we put*

$$F^c(x) := \int_0^\infty \mathbb{1}_{(x,\infty)}(y)\, c(dy), \tag{4.1}$$

*If $c \in \mathcal{M}_\lambda^+$ for some $\lambda \in (0,1]$, then*

$$\int_0^\infty x^{\lambda-1} F^c(x)\, dx = M_\lambda(c)/\lambda, \qquad \lim_{x\to 0} x^\lambda F^c(x) = \lim_{x\to\infty} x^\lambda F^c(x) = 0,$$

*and $F^c \in L^\infty(\varepsilon,\infty)$ for each $\varepsilon > 0$.*

We give now a very important inequality on which the existence and uniqueness proof relies.

**Proposition 4.3.** *Consider $\lambda \in (0,1]$, a coagulation kernel $K$, a fragmentation kernel $F$ and a measure $\beta$ on $\Theta$ satisfying Hypotheses 2.1. and 2.2. with the same $\lambda$. Let $c^{in}$ and $d^{in} \in \mathcal{M}_\lambda^+$ and denote by $(c_t)_{t\in[0,\infty)}$ a $(c^{in}, K, F, \beta, \lambda)$-weak solution to (2.12) and by $(d_t)_{t\in[0,\infty)}$ a $(d^{in}, K, F, \beta, \lambda)$-weak solution to (2.12). In addition, we put $E(t,x) = F^{c_t}(x) - F^{d_t}(x)$, $\rho(x) = x^{\lambda-1}$ and*

$$R(t,x) = \int_0^x \rho(z) sign(E(t,z))\, dz \ \text{ for } (t,x) \in [0,\infty)\times(0,\infty).$$

*Then, for each $t \in [0,\infty)$, $R(t,\cdot) \in \mathcal{H}_\lambda$ and (recall (4.1) and (2.10))*

$$\begin{aligned} \frac{d}{dt} d_\lambda(c_t,d_t) &= \frac{d}{dt}\int_0^\infty x^{\lambda-1}|E(t,x)|dx \\ &\le \frac{1}{2}\int_0^\infty\int_0^\infty K(x,y)\,[\rho(x+y)-\rho(x)]\,(c_t+d_t)(dy)|E(t,x)|\,dx \\ &\quad + \frac{1}{2}\int_0^\infty\int_0^\infty \partial_x K(x,y)\,(AR(t))\,(x,y)(c_t+d_t)(dy)E(t,x)\,dx \\ &\quad + \int_0^\infty F'(x)\int_\Theta (BR(t))(\theta,x)\beta(d\theta)E(t,x)dx \\ &\quad + \int_0^\infty F(x)x^{\lambda-1}|E(t,x)|\int_\Theta \left(\sum_{i\ge1}\theta_i^\lambda - 1\right)\beta(d\theta)dx. \end{aligned} \tag{4.2}$$

Note that it is straightforward that under the notation and assumptions of Proposition 4.3., from (2.4), (2.5), (2.9) and using Lemma 4.4. below, there exists a positive constant $C_1$ depending on $\lambda$, $\kappa_0$ and $\kappa_1$ and a positive constant $C_2$ depending on $\kappa_2$, $\kappa_3$ and $C_\beta^\lambda$ such that for each $t \in [0,\infty)$,

$$\frac{d}{dt} d_\lambda(c_t,d_t) \le (C_1 M_\lambda(c_t+d_t) + C_2)\, d_\lambda(c_t,d_t). \tag{4.3}$$

Before to give the proof of Proposition 4.3., we state two auxiliary results. In Lemma 4.4. are given some inequalities which are useful to verify that the integrals on the right-hand side of (4.2) are convergent, and in Lemma 4.5. we study the time differentiability of $E$.

**Lemma 4.4.** *Under the notation and assumptions of Proposition 4.3., there exists a positive constant $C$ such that for $(t,x,y) \in [0,\infty)\times(0,\infty)^2$,*

$$\begin{aligned} K(x,y)\,|\rho(x+y)-\rho(x)| &\le Cx^{\lambda-1}y^\lambda, \\ K(x,y)\,|(AR(t))\,(x,y)| &\le Cx^\lambda y^\lambda, \\ |\partial_x K(x,y)\,(AR(t))\,(x,y)| &\le Cx^{\lambda-1}y^\lambda, \\ \int_\Theta |(BR(t))(\theta,x)|\,\beta(d\theta) &\le CC_\beta^\lambda x^\lambda. \end{aligned} \tag{4.4}$$

*Proof.* The first three inequalities were proved in [15, Lemma 3.4]. In particular, recall that

$$|(AR(t))\,(x,y)| \le \frac{2}{\lambda}(x\wedge y)^{\lambda}, \tag{4.5}$$

for $(t,x,y)\in[0,\infty)\times(0,\infty)^2$. Next, using (2.9) we deduce

$$\begin{aligned}\int_\Theta |(BR(t))(\theta,x)|\,\beta(d\theta) &= \left|\int_\Theta\left[\sum_{i\ge1} R(t,\theta_i x)-R(t,x)\right]\beta(d\theta)\right|\\ &= \int_\Theta\left|\sum_{i\ge2}\int_0^{\theta_i x}\partial_x R(t,z)dz-\int_{\theta_1 x}^{x}\partial_x R(t,z)dz\right|\beta(d\theta)\\ &\le \int_\Theta\left(\sum_{i\ge2}\int_0^{\theta_i x} z^{\lambda-1}dz+\int_{\theta_1 x}^{x} z^{\lambda-1}dz\right)\beta(d\theta) \;\le\; \frac{1}{\lambda}C_\beta^\lambda x^\lambda.\end{aligned}$$

□

**Lemma 4.5.** *Consider $\lambda\in(0,1]$, a coagulation kernel $K$, a fragmentation kernel $F$ and a measure $\beta$ on $\Theta$ satisfying the Hypotheses 2.1. with the same $\lambda$. Let $c^{in}\in\mathcal{M}_\lambda^+$ and denote by $(c_t)_{t\in[0,\infty)}$ a $(c^{in},K,F,\beta,\lambda)$-weak solution to (2.12). Then*

*$(x,t)\mapsto \partial_t F^{c_t}(x)$ belongs to $L^\infty(0,s;L^1(0,\infty;x^{\lambda-1}dx))$, for each $s\in[0,\infty)$.*

*Proof.* Following the same ideas as in [15], we consider $\vartheta\in\mathcal{C}([0,\infty))$ with compact support in $(0,\infty)$, we put

$$\phi(x)=\int_0^x \vartheta(y)\,dy, \qquad \text{for } x\in(0,\infty),$$

this function belongs to $\mathcal{H}_\lambda$. First, performing an integration by parts and using Lemma 4.2. we obtain

$$\int_0^\infty \vartheta(x)F^{c_t}(x)\,dx = \int_0^\infty \phi(x)c_t(dx).$$

Next, on the one hand recall that in [15, eq. (3.7)] was proved that

$$\begin{aligned}&\int_0^\infty\int_0^\infty K(x,y)\,(A\phi)\,(x,y)\,c_t(dy)\,c_t(dx)dz\\ &\qquad= \int_0^\infty \vartheta(z)\int_0^z\int_0^z \mathbb{1}_{[z,\infty)}(x+y)K(x,y)c_t(dy)\,c_t(dx)dz\\ &\qquad\quad-\int_0^\infty \vartheta(z)\int_z^\infty\int_z^\infty K(x,y)c_t(dy)\,c_t(dx)dz.\end{aligned}$$

On the other hand, using the Fubini Theorem, we have

$$\begin{aligned}&\int_0^\infty F(x)\int_\Theta (B\phi)\,(\theta,x)\beta(d\theta)c_t(dx)\\ &\qquad= \int_0^\infty F(x)\int_\Theta\left[\sum_{i\ge1}\int_0^{\theta_i x}\vartheta(z)dz-\int_0^x\vartheta(z)dz\right]\beta(d\theta)\,c_t(dx)\\ &\qquad= \int_0^\infty \vartheta(z)\int_\Theta\left[\sum_{i\ge1}\int_{z/\theta_i}^\infty F(x)c_t(dx)-\int_z^\infty F(x)c_t(dx)\right]\beta(d\theta)\,dz.\end{aligned}$$

Thus, from (2.12) we infer that

$$\begin{aligned}\frac{d}{dt}\int_0^\infty \vartheta(x)F^{c_t}(x)\,dx &= \frac{1}{2}\int_0^\infty \vartheta(z)\int_0^z\int_0^z \mathbb{1}_{[z,\infty)}(x+y)K(x,y)c_t(dy)\,c_t(dx)dz\\ &\quad -\frac{1}{2}\int_0^\infty \vartheta(z)\int_z^\infty\int_z^\infty K(x,y)c_t(dy)\,c_t(dx)dz\\ &\quad +\int_0^\infty \vartheta(z)\int_\Theta\left[\sum_{i\geq 1}\int_{z/\theta_i}^\infty F(x)c_t(dx)-\int_z^\infty F(x)c_t(dx)\right]\beta(d\theta)\,dz,\end{aligned}$$

whence

$$\begin{aligned}\partial_t F^{c_t}(z) &= \frac{1}{2}\int_0^z\int_0^z \mathbb{1}_{[z,\infty)}(x+y)K(x,y)c_t(dy)\,c_t(dx)-\frac{1}{2}\int_z^\infty\int_z^\infty K(x,y)c_t(dy)\,c_t(dx)\\ &\quad +\int_\Theta\left[\sum_{i\geq 1}\int_{z/\theta_i}^\infty F(x)c_t(dx)\beta(d\theta)-\int_z^\infty F(x)c_t(dx)\right]\beta(d\theta),\end{aligned} \tag{4.6}$$

for $(t,z)\in[0,\infty)\times(0,\infty)$. First, in [15, Lemma 3.5] it was shown that,

$$\begin{aligned}\int_0^\infty z^{\lambda-1}\left|\frac{1}{2}\int_0^z\int_0^z \mathbb{1}_{[z,\infty)}(x+y)K(x,y)c_t(dy)\,c_t(dx)-\frac{1}{2}\int_z^\infty\int_z^\infty K(x,y)c_t(dy)\,c_t(dx)\right|dz&\\ \leq \frac{2\kappa_0}{\lambda}M_\lambda(c_t)^2.&\end{aligned}$$

Thus, from (2.4) and the Fubini Theorem follows that, for each $t\in[0,\infty)$,

$$\begin{aligned}\int_0^\infty z^{\lambda-1}|\partial F^{c_t}(z)|dz &\leq \frac{2\kappa_0}{\lambda}M_\lambda(c_t)^2\\ &\quad +\int_0^\infty z^{\lambda-1}\left|\int_\Theta\left(\sum_{i\geq 2}\int_{z/\theta_i}^\infty F(x)c_t(dx)-\int_z^{z/\theta_1} F(x)c_t(dx)\right)\right|\beta(d\theta)\,dz\\ &\leq \frac{2\kappa_0}{\lambda}M_\lambda(c_t)^2+\kappa_2\int_\Theta\int_0^\infty\left(\sum_{i\geq 2}\int_0^{\theta_i x} z^{\lambda-1}dz+\int_{\theta_1 x}^x z^{\lambda-1}dz\right)c_t(dx)\beta(d\theta)\\ &\leq \frac{2\kappa_0}{\lambda}M_\lambda(c_t)^2+\frac{\kappa_2}{\lambda}M_\lambda(c_t)\left[\int_\Theta\left(\sum_{i\geq 2}\theta_i^\lambda+(1-\theta_1^\lambda)\right)\beta(d\theta)\right]\\ &\leq \frac{2\kappa_0}{\lambda}M_\lambda(c_t)^2+\frac{C_\beta^\lambda\kappa_2}{\lambda}M_\lambda(c_t),\end{aligned}$$

where we have used (2.7). Finally, since the right-hand side of the above inequality is bounded on $[0,t]$ for all $t>0$ by (2.11), we obtain the expected result. □

*Proof of Proposition 4.3.* Let $t\in[0,\infty)$. We first note that, since $s\mapsto M_\lambda(c_s)$ and $s\mapsto M_\lambda(d_s)$ are in $L^\infty(0,t)$ by (2.11), it follows from Lemmas 4.2. and 4.4. that the integrals in (4.2) are absolutely convergent. Furthermore, for $t \geq 0$ and $x > y$, we have

$$\begin{aligned} |R(t,x) - R(t,y)| &= \left| \int_y^x z^{\lambda-1} sign(E(t,z))\, dz \right| \\ &\leq \frac{1}{\lambda}(x^\lambda - y^\lambda) = \frac{1}{\lambda}\left((x-y+y)^\lambda - y^\lambda\right) \\ &\leq \frac{1}{\lambda}(x-y)^\lambda, \end{aligned}$$

since $\lambda \in (0,1]$. Thus $R(t,\cdot) \in \mathcal{H}_\lambda$ for each $t \in [0,\infty)$.

Next, by Lemmas 4.2 and 4.5, $E \in W^{1,\infty}(0,s;L^1(0,\infty;x^{\lambda-1}dx))$ for every $s \in (0,T)$, so that

$$\begin{aligned} \frac{d}{dt}\int_0^\infty x^{\lambda-1}|E(t,x)|dx &= \int_0^\infty x^{\lambda-1} sign(E(t,x))\, \partial_t E(t,x)\, dx \\ &= \int_0^\infty \partial_x R(t,x) \left(\partial_t F^{c_t}(x) - \partial_t F^{d_t}(x)\right) dx. \end{aligned}$$

We use (4.6) to obtain

$$\begin{aligned} &\frac{d}{dt}\int_0^\infty x^{\lambda-1}|E(t,x)|dx \\ &\quad = \frac{1}{2}\int_0^\infty \partial_x R(t,z) \int_0^z \int_0^z \mathbb{1}_{[z,\infty)}(x+y) K(x,y)(c_t(dy)\, c_t(dx) - d_t(dy)\, d_t(dx))\, dz \\ &\qquad -\frac{1}{2}\int_0^\infty \partial_x R(t,z) \int_z^\infty \int_z^\infty K(x,y)(c_t(dy)\, c_t(dx) - d_t(dy)\, d_t(dx))\, dz \\ &\qquad + \int_0^\infty \partial_x R(t,z) \int_\Theta \left[ \sum_{i\geq 1} \int_{z/\theta_i}^\infty F(x)(c_t - d_t)(dx) - \int_z^\infty F(x)(c_t-d_t)(dx) \right] \beta(d\theta) dz. \end{aligned} \tag{4.7}$$

Recalling [15, eq. (3.8)] and using the Fubini Theorem we obtain

$$\frac{d}{dt}\int_0^\infty x^{\lambda-1}|E(t,x)|dx = \frac{1}{2}\int_0^\infty I^c(t,x)\,(c_t - d_t)\,(dx) + \int_0^\infty I^f(t,x)\,(c_t-d_t)\,(dx), \tag{4.8}$$

where

$$\begin{aligned} I^c(t,x) &= \int_0^\infty K(x,y)(AR(t))(x,y)(c_t+d_t)(dy), & x \in (0,\infty) \\ I^f(t,x) &= F(x)\int_\Theta (BR(t))(\theta,x)\beta(d\theta), & x \in (0,\infty). \end{aligned}$$

It follows from (4.4) with (2.4) that

$$|I^f(t,x)| \leq C\, x^\lambda, \;\; x \in (0,\infty), \;\; t \in [0,\infty). \tag{4.9}$$

We would like to be able to perform an integration by parts in the second integral of the right hand of (4.8). However, $I^f$ is not necessarily differentiable with respect to $x$. We thus fix $\varepsilon \in (0,1)$ and put

$$I^f_\varepsilon(t,x) = F(x)\int_\Theta (BR(t))(\theta,x)\beta_\varepsilon(d\theta), \qquad x \in (0,\infty),$$

where $\beta_\varepsilon$ is the finite measure $\beta|_{\Theta_\varepsilon}$ with $\Theta_\varepsilon = \{\theta \in \Theta : \theta_1 \leq 1-\varepsilon\}$ and note that

$$\beta_\varepsilon(\Theta) = \int_\Theta \mathbb{1}_{\{1-\theta_1 \geq \varepsilon\}}\, \beta(d\theta) \leq \frac{1}{\varepsilon}\int_\Theta (1-\theta_1)\, \beta(d\theta) \leq \frac{1}{\varepsilon} C^\lambda_\beta < \infty. \tag{4.10}$$

Since $F$ belongs to $W^{1,\infty}(\alpha, 1/\alpha)$ for $\alpha \in (0,1)$ and $|R(t,x)| \leq x^\lambda/\lambda$ and $|\partial_x R(t,x)| \leq x^{\lambda-1}$ we deduce that $I^f_\varepsilon \in W^{1,\infty}(\alpha, 1/\alpha)$ for $\alpha \in (0,1)$ with

$$\partial_x I^f_\varepsilon(t,x) = F'(x)\int_\Theta (BR(t))(\theta,x)\beta_\varepsilon(d\theta) + F(x)\int_\Theta \left[\sum_{i\geq 1}\theta_i \partial_x R(t,\theta_i x) - \partial_x R(t,x)\right]\beta_\varepsilon(d\theta). \tag{4.11}$$

We now perform an integration by parts to obtain

$$\begin{aligned}\int_0^\infty I^f(t,x)\,(c_t-d_t)\,(dx) &= \int_0^\infty \left(I^f - I^f_\varepsilon\right)(t,x)\,(c_t-d_t)\,(dx) - [I^f_\varepsilon(t,x)E(t,x)]_{x=0}^{x=\infty} \\ &\quad + \int_0^\infty \partial_x I^f_\varepsilon(t,x)E(t,x)\,dx.\end{aligned} \tag{4.12}$$

First, we have

$$\begin{aligned}\left|\int_0^\infty \left(I^f - I^f_\varepsilon\right)(t,x)\,(c_t-d_t)\,(dx)\right| &\leq \int_0^\infty \left|\left(I^f - I^f_\varepsilon\right)(t,x)\right|(c_t+d_t)\,(dx) \\ &\leq \kappa_2 \int_0^\infty \int_\Theta |(BR(t))(\theta,x)|\,(\beta-\beta_\varepsilon)(d\theta)(c_t+d_t)(dx) \\ &\leq \kappa_2 \int_0^\infty \int_\Theta \left(\sum_{i\geq 2}\int_0^{\theta_i x} z^{\lambda-1}dz + \int_{\theta_1 x}^x z^{\lambda-1}dz\right)\mathbb{1}_{\{1-\theta_1<\varepsilon\}}\beta(d\theta)(c_t+d_t)(dx) \\ &\leq \frac{\kappa_2}{\lambda}\int_0^\infty x^\lambda \int_\Theta \left[\sum_{i\geq 2}\theta_i^\lambda + (1-\theta_1)^\lambda\right]\mathbb{1}_{\{1-\theta_1<\varepsilon\}}\beta(d\theta)(c_t+d_t)(dx) \\ &= \frac{\kappa_2}{\lambda}M_\lambda(c_t+d_t)\int_\Theta \left[\sum_{i\geq 2}\theta_i^\lambda + (1-\theta_1)^\lambda\right]\mathbb{1}_{\{1-\theta_1<\varepsilon\}}\beta(d\theta),\end{aligned}$$

whence, recalling (2.7)

$$\lim_{\varepsilon\to 0}\int_0^\infty \left(I^f - I^f_\varepsilon\right)(t,x)\,(c_t-d_t)\,(dx) = 0. \tag{4.13}$$

Next, it follows from (4.9) that

$$|I^f_\varepsilon(t,x)E(t,x)| \leq Cx^\lambda\left(F^{c_t}(x) + F^{d_t}(x)\right),\ \ x\in(0,\infty),\ \ t\in[0,\infty),$$

we can thus easily conclude by Lemma 4.2. that

$$\lim_{x\to 0} I^f_\varepsilon(t,x)E(t,x) = \lim_{x\to\infty} I^f_\varepsilon(t,x)E(t,x) = 0. \tag{4.14}$$

Finally, (2.5), Lemma 4.2. and (4.4) imply that

$$\lim_{\varepsilon\to 0}\int_0^\infty F'(x)\int_\Theta (BR(t))(\theta,x)\beta_\varepsilon(d\theta)E(t,x)dx = \int_0^\infty F'(x)\int_\Theta (BR(t))(\theta,x)\beta(d\theta)E(t,x)dx, \tag{4.15}$$

while

(4.16)

$$\begin{aligned}
&\limsup_{\varepsilon\to 0}\int_0^\infty F(x)\int_\Theta\left[\sum_{i\geq 1}\theta_i\partial_x R(t,\theta_i x)-\partial_x R(t,x)\right]\beta_\varepsilon(d\theta)E(t,x)dx\\
&=\limsup_{\varepsilon\to 0}\int_0^\infty F(x)\int_\Theta\left(\sum_{i\geq 1}\theta_i^\lambda x^{\lambda-1}sign(E(t,\theta_i x))-x^{\lambda-1}sign(E(t,x))\right)\beta_\varepsilon(d\theta)E(t,x)dx\\
&=\limsup_{\varepsilon\to 0}\int_0^\infty F(x)x^{\lambda-1}sign(E(t,x))E(t,x)\int_\Theta\left(\sum_{i\geq 1}\theta_i^\lambda sign\left(E(t,\theta_i x)E(t,x)\right)-1\right)\beta_\varepsilon(d\theta)dx\\
&\leq\limsup_{\varepsilon\to 0}\int_0^\infty F(x)x^{\lambda-1}|E(t,x)|\int_\Theta\left(\sum_{i\geq 1}\theta_i^\lambda-1\right)\beta_\varepsilon(d\theta)dx\\
&=\int_0^\infty F(x)x^{\lambda-1}|E(t,x)|\int_\Theta\left(\sum_{i\geq 1}\theta_i^\lambda-1\right)\beta(d\theta)dx.
\end{aligned}$$

We have used (2.9) and (2.7). Note that we are only interested in an upper bound, when the term $\sum_{i\geq 1}\theta_i^\lambda-1$ is negative, 0 would be a better bound for the last term.

Recall (4.8), the term involving $I^c$ was treated in [15, Proposition 3.3], while from (4.12) with (4.13), (4.14), (4.15) and (4.16) we deduce the inequality (4.2), which completes the proof of Proposition 4.3. □

4.1. **Proof of Theorem 2.5.**

*Uniqueness.* Owing to (2.11) and (4.3), the uniqueness assertion of Theorem 2.5. readily follows from the Gronwall Lemma. □

*Existence.* The proof of the existence assertion of Theorem 2.5. is split into three steps. The first step consists in finding an approximation to the coagulation-fragmentation equation by a version of (2.12) with finite operators: we will show existence in the set of positive measures with finite total variation, i.e. $\mathcal{M}_0^+$, using the Picard method.

Next, we will show existence of a weak solution to (1.1) with an initial condition $c^{in}$ in $\mathcal{M}_\lambda^+\cap\mathcal{M}_2^+$, the final step consists in extending this result to the case where $c^{in}$ belongs only to $\mathcal{M}_\lambda^+$.

**Bounded Case : existence and uniqueness in $\mathcal{M}_0^+$.-**

We consider a bounded coagulation kernel and a fragmentation mechanism which gives only a finite number of fragments. This is

$$\left\{\begin{array}{rcll} K(x,y) & \leq & \overline{K}, & \text{for some } \overline{K}\in\mathbb{R}^+\\ F(x) & \leq & \overline{F}, & \text{for some } \overline{F}\in\mathbb{R}^+\\ \beta(\Theta) & < & \infty, & \\ \beta(\Theta\setminus\Theta_k) & = & 0, & \text{for some } k\in\mathbb{N}, \end{array}\right. \tag{4.17}$$

where

$$\Theta_k=\left\{\theta=(\theta_n)_{n\geq 1}\in\Theta:\,\theta_{k+1}=\theta_{k+2}=\ldots=0\right\}.$$

We will show in this paragraph that under this assumptions there exists a global weak-solution to (1.1). We will use the notation $\|\cdot\|_\infty$ for the sup norm on $L^\infty[0,\infty)$ and $\|\cdot\|_{VT}$ for the total variation norm on measures. The result reads as follows.

**Proposition 4.6.** *Consider $\mu^{in} \in \mathcal{M}_0^+$. Assume that the coagulation and fragmentation kernels $K$ and $F$ and the measure $\beta$ satisfy the assumptions (4.17). Then, there exists a unique non-negative weak-solution $(\mu_t)_{t\geq 0}$ starting at $\mu_0 = \mu^{in}$ to (1.1). Furthermore, it satisfies for all $t \geq 0$,*

$$\sup_{[0,t]} \|\mu_s\|_{VT} \leq C_t \, \|\mu^{in}\|_{VT}, \tag{4.18}$$

*where $C_t$ is a positive constant depending on $t$, $\overline{K}$, $\overline{F}$ and $\beta$.*

**Remark 4.7.** *Proposition 4.6. deals with weak solutions to (1.1) with $\mu^{in} \in \mathcal{M}_0^+$ and with respect to the set of test functions $\phi \in L^\infty([0,\infty))$. However, when $\mu^{in} \in \mathcal{M}_\lambda^+$, we can apply equation (2.12) with $\phi(x) = x^\lambda \wedge A$ with $A > 0$, the Gronwall Lemma and then make tend $A$ to infinity to prove that*

$$\sup_{[0,T]} M_\lambda(\mu_t) < \infty, \ \forall T \geq 0.$$

*In the same way, using this last bound together with (4.17), (4.18) and the Lebesgue dominated convergence Theorem, we extend readily to $\phi \in \mathcal{H}_\lambda$. Hence, whenever $\mu^{in} \in \mathcal{M}_\lambda^+$ we obtain a $(\mu^{in}, K, F, \beta, \lambda)$-weak solution $(\mu_t)_{t\geq 0}$ to (2.12).*

To prove this proposition we need to replace the operator $A$ in (2.12) by an equivalent one, this new operator will be easier to manipulate. We consider, for $\phi$ a bounded function, the following operators

$$(\tilde{A}\phi)(x,y) = K(x,y)\left[\frac{1}{2}\phi(x+y) - \phi(x)\right], \tag{4.19}$$

$$(L\phi)(x) = F(x)\int_\Theta \left(\sum_{i\geq 1}\phi(\theta_i x) - \phi(x)\right)\beta(d\theta). \tag{4.20}$$

Thus, (2.12) can be rewritten as

$$\frac{d}{dt}\int_0^\infty \phi(x)c_t(dx) = \int_0^\infty \left[\int_0^\infty (\tilde{A}\phi)(x,y)c_t(dy) + (L\phi)(x)\right] c_t(dx). \tag{4.21}$$

The Proposition will be proved using an implicit scheme for equation (4.21). First, we need to provide a unique and non-negative solution to this scheme.

**Lemma 4.8.** *Consider $\mu^{in} \in \mathcal{M}_0^+$ and let $(\nu_t)_{t\geq 0}$ be a family of measures in $\mathcal{M}_0^+$ such that $\sup_{[0,t]} \|\nu_s\|_{VT} < \infty$ for all $t \geq 0$. Then, under the assumptions (4.17), there exists a unique non-negative solution $(\mu_t)_{t\geq 0}$ starting at $\mu_0 = \mu^{in}$ to*

$$\int_0^\infty \phi(x)\mu_t(dx) = \int_0^\infty \phi(x)\mu_0(dx) + \int_0^t\int_0^\infty \left[\int_0^\infty (\tilde{A}\phi)(x,y)\nu_s(dy) + (L\phi)(x)\right]\mu_s(dx)ds \tag{4.22}$$

*for all $\phi \in L^\infty(\mathbb{R}^+)$. Furthermore, the solution satisfies for all $t \geq 0$,*

$$\sup_{[0,t]} \|\mu_s\|_{VT} \leq C_t \, \|\mu^{in}\|_{VT}, \tag{4.23}$$

*where $C_t$ is a positive constant depending on $t$, $\overline{K}$, $\overline{F}$ and $\beta$.*

*The constant $C_t$ does not depend on* $\sup_{[0,t]} \|\nu_s\|_{VT}$.

We will prove this lemma in two steps. First, we show that (4.22) is equivalent to another equation. This new equation is constructed in such a way that the negative terms of equation (4.22) are eliminated. Next, we prove existence and uniqueness for this new equation. This solution will be proved to be non-negative and it will imply existence, uniqueness and non-negativity of a solution to (4.22).

*Proof.* **Step 1.-** First, we give now an auxiliary result which allows to differentiate equation (4.25) when the test function depends on $t$.

**Lemma 4.9.** *Let $(t,x) \mapsto \phi_t(x) : \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}$ be a bounded measurable function, having a bounded partial derivative $\partial\phi/\partial t$ and consider $(\mu_t)_{t\geq 0}$ a weak-solution to (4.22). Then, for all $t \geq 0$,*

$$\frac{d}{dt}\int_0^\infty \phi_t(x)\mu_t(dx) = \int_0^\infty \frac{\partial}{\partial t}\phi_t(x)\mu_t(dx) + \int_0^\infty\int_0^\infty (\tilde{A}\phi_t)(x,y)\mu_t(dx)\nu_t(dy) + \int_0^\infty (L\phi_t)(x)\mu_t(dx).$$

*Proof.* First, note that for $0 \leq t_1 \leq t_2$ we have,

$$\begin{aligned}
&\int_0^\infty \phi_{t_2}(x)\mu_{t_2}(dx) - \int_0^\infty \phi_{t_1}(x)\mu_{t_1}(dx) \\
&\quad= \int_0^\infty \left(\phi_{t_2}(x) - \phi_{t_1}(x)\right)\mu_{t_2}(dx) + \int_0^\infty \phi_{t_1}(x)\left(\mu_{t_2} - \mu_{t_1}\right)(dx) \\
&\quad= \int_{t_1}^{t_2}\int_0^\infty \frac{\partial}{\partial t}\phi_s(x)\mu_{t_2}(dx)ds + \int_{t_1}^{t_2}\frac{d}{dt}\int_0^\infty \phi_{t_1}(x)\mu_t(dx)dt \\
&\quad= \int_{t_1}^{t_2}\int_0^\infty \frac{\partial}{\partial t}\phi_s(x)\mu_{t_2}(dx)ds \\
&\qquad+ \int_{t_1}^{t_2}\left[\int_0^\infty\int_0^\infty (\tilde{A}\phi_{t_1})(x,y)\mu_s(dx)\nu_s(dy) + \int_0^\infty (L\phi_{t_1})(x)\mu_s(dx)\right]ds.
\end{aligned}$$

Thus, fix $t > 0$ and set for $n \in \mathbb{N}$, $t_k = t\dfrac{k}{n}$ with $k = 0, 1, \ldots, n$, we get

$$\begin{aligned}
\int_0^\infty \phi_t(x)\mu_t(dx) &= \int_0^\infty \phi_0(x)\mu_0(dx) + \sum_{k=1}^n\left[\int_0^\infty \phi_{t_k}(x)\mu_{t_k}(dx) - \int_0^\infty \phi_{t_{k-1}}(x)\mu_{t_{k-1}}(dx)\right] \\
&= \int_0^\infty \phi_0(x)\mu_0(dx) + \sum_{k=1}^n\int_{t_{k-1}}^{t_k}\int_0^\infty \frac{\partial}{\partial t}\phi_s(x)\mu_{t_k}(dx)ds \\
&\quad+ \sum_{k=1}^n\int_{t_{k-1}}^{t_k}\left[\int_0^\infty\int_0^\infty (\tilde{A}\phi_{t_{k-1}})(x,y)\mu_s(dx)\nu_s(dy)s + \int_0^\infty (L\phi_{t_{k-1}})(x)\mu_s(dx)\right]ds.
\end{aligned}$$

Next, for $s \in [t_{k-1}, t_k)$ we set $k = \left\lfloor \dfrac{ns}{t} \right\rfloor$ and use the notation $\overline{s}_n := t_k = \dfrac{t}{n}\left\lfloor \dfrac{ns}{t} \right\rfloor$ and $\underline{s}_n := t_{k-1}$. Thus, the equation above can be rewritten as

$$\begin{aligned}
\int_0^\infty \phi_t(x)\mu_t(dx) &= \int_0^\infty \phi_0(x)\mu_0(dx) + \int_0^t\int_0^\infty \frac{\partial}{\partial t}\phi_s(x)\mu_{\overline{s}_n}(dx)ds \\
&\quad+ \int_0^t\int_0^\infty\int_0^\infty (\tilde{A}\phi_{\underline{s}_n})(x,y)\mu_s(dx)\nu_s(dy)ds + \int_0^t\int_0^\infty (L\phi_{\underline{s}_n})(x)\mu_s(dx)ds,
\end{aligned}$$

and the lemma follows from letting $n \to \infty$ since $\overline{s}_n \to s$. □

Next, we introduce a new equation. We put for $t \geq 0$,

$$\gamma_t(x) = \exp\left[\int_0^t \left(\int_0^\infty K(x,y)\nu_s(dy) - F(x)\right) ds\right], \tag{4.24}$$

and we consider the equation

$$\begin{aligned}\frac{d}{dt}\int_0^\infty \phi(x)\tilde{\mu}_t(dx) \quad = \quad & \int_0^\infty \Bigg[\int_0^\infty \frac{1}{2}K(x,y)(\phi\gamma_t)(x+y)\nu_t(dy) \\ & + F(x)\int_\Theta \sum_{i\geq 1}(\phi\gamma_t)(\theta_i x)\beta(d\theta)\Bigg]\gamma_t^{-1}(x)\tilde{\mu}_t(dx).\end{aligned} \tag{4.25}$$

Now, we give a result that relates (4.22) to (4.25).

**Lemma 4.10.** *Consider $\mu^{in} \in \mathcal{M}_0^+$ and recall (4.24). Then, $(\mu_t)_{t\geq 0}$ with $\mu_0 = \mu^{in}$ is a weak-solution to (4.22) if and only if $(\tilde{\mu}_t)_{t\geq 0}$ with $\tilde{\mu}_0 = \mu^{in}$ is a weak-solution to (4.25), where $\tilde{\mu}_t = \gamma_t\mu_t$ for all $t \geq 0$.*

*Proof.* First, assume that $(\mu_t)_{t\geq 0}$ is a weak-solution to (4.22).

We have $\dfrac{\partial}{\partial t}\gamma_t(x) = \gamma_t(x)\left[\int_0^\infty K(x,y)\nu_t(dy) - F(x)\right]$. Note that $\gamma_t$, $\gamma_t^{-1}$ and $\frac{\partial}{\partial t}\gamma_t$ are bounded on $[0,t]$ for all $t \geq 0$, by (4.17) and since $\sup_{[0,t]}\|\nu_s\|_{VT} < \infty$.

Set $\tilde{\mu}_t = \gamma_t\mu_t$, recall (4.19) and (4.20), by Lemma 4.9., for all bounded measurable functions $\phi$, we have

$$\begin{aligned}\frac{d}{dt}\int_0^\infty \phi(x)\tilde{\mu}_t(dx) &= \int_0^\infty \phi(x)\gamma_t(x)\left[\int_0^\infty K(x,y)\nu_t(dy) - F(x)\right]\mu_t(dx) \\ &\quad + \int_0^\infty\int_0^\infty \left[\frac{1}{2}(\phi\gamma_t)(x+y) - (\phi\gamma_t)(x)\right]K(x,y)\nu_t(dy)\mu_t(dx) \\ &\quad + \int_0^\infty F(x)\int_\Theta\left(\sum_{i\geq 1}(\phi\gamma_t)(\theta_i x) - (\phi\gamma_t)(x)\right)\beta(d\theta)\mu_t(dx) \\ &= \int_0^\infty\int_0^\infty \frac{1}{2}K(x,y)(\phi\gamma_t)(x+y)\nu_t(dy)\mu_t(dx) \\ &\quad + \int_0^\infty F(x)\int_\Theta\sum_{i\geq 1}(\phi\gamma_t)(\theta_i x)\beta(d\theta)\mu_t(dx) \\ &= \int_0^\infty \Bigg[\int_0^\infty \frac{1}{2}K(x,y)(\phi\gamma_t)(x+y)\nu_t(dy) \\ &\qquad + F(x)\int_\Theta\sum_{i\geq 1}(\phi\gamma_t)(\theta_i x)\beta(d\theta)\Bigg]\gamma_t^{-1}(x)\tilde{\mu}_t(dx),\end{aligned}$$

and the result follows.

For the reciprocal assertion, we assume that $(\tilde{\mu}_t)_{t\geq 0}$ is a weak-solution to (4.25), set $\mu_t = \gamma_t^{-1}\tilde{\mu}_t$ and we show in the same way that $(\mu_t)_{t\geq 0}$ is a weak-solution to (4.22). □

We note that, since all the terms between the brackets are non-negative, the right-hand side of equation (4.25) is non-negative whenever $\tilde{\mu}_t \geq 0$. Thus, $\gamma_t$ is an integrating factor that removes the negative terms of equation (4.22).

**Step 2.-** We define the following explicit scheme for (4.25): we set $\tilde{\mu}_t^0 = \mu^{in}$ for all $t \geq 0$ and for $n \geq 0$

(4.26)

$$\begin{cases} \frac{d}{dt}\int_0^\infty \phi(x)\tilde{\mu}_t^{n+1}(dx) &= \int_0^\infty \left[\int_0^\infty \frac{1}{2}K(x,y)(\phi\gamma_t)(x+y)\nu_t(dy) \right. \\ & \qquad \left. +F(x)\int_\Theta \sum_{i\geq 1}(\phi\gamma_t)(\theta_i x)\beta(d\theta)\right]\gamma_t^{-1}(x)\tilde{\mu}_t^n(dx) \\ \tilde{\mu}_0^{n+1} &= \mu^{in}. \end{cases}$$

Recall (4.17), note that the following operators are bounded:

$$\left\|\gamma_t^{-1}(\cdot)\int_0^\infty \frac{1}{2}K(\,\cdot\,,y)(\phi\gamma_t)(\,\cdot+y)\nu_t(dy)\right\|_\infty \leq C_t\|\phi\|_\infty, \tag{4.27}$$

$$\left\|\gamma_t^{-1}(\cdot)F(\cdot)\int_\Theta \sum_{i\geq 1}(\phi\gamma_t)(\theta_i\,\cdot)\beta(d\theta)\right\|_\infty \leq C_t\|\phi\|_\infty, \tag{4.28}$$

where $C_t$ is a positive constant depending on $\overline{K}$, $\overline{F}$, $\beta$ and $\sup_{[0,t]}\|\nu_s\|_{VT}$.

Thus, we consider $\phi$ bounded, integrate in time (4.26), use (4.27) and (4.28) to obtain

$$\begin{aligned} \int_0^\infty \phi(x)\left(\tilde{\mu}_t^{n+1}(dx) - \tilde{\mu}_t^n(dx)\right) &\leq C_{1,t}\|\phi\|_\infty \int_0^t \left\|\tilde{\mu}_s^n - \tilde{\mu}_s^{n-1}\right\|_{VT} ds \\ &\quad + C_{2,t}\|\phi\|_\infty \int_0^t \left\|\tilde{\mu}_s^n - \tilde{\mu}_s^{n-1}\right\|_{VT} ds, \end{aligned}$$

note that the the difference of the initial conditions vanishes since they are the same. We take the sup over $\|\phi\|_\infty \leq 1$ and use $\sup_{[0,t]}\|\nu_s\|_{VT} < \infty$ to deduce

$$\left\|\tilde{\mu}_t^{n+1} - \tilde{\mu}_t^n\right\|_{VT} \leq C_t \int_0^t \left\|\tilde{\mu}_s^n - \tilde{\mu}_s^{n-1}\right\|_{VT} ds,$$

where $C_t$ is a positive constant depending on $\overline{K}$, $\overline{F}$, $\beta$, $\sup_{[0,t]}\|\nu_s\|_{VT}$ and $\|\phi\|_\infty$. Hence, by classical arguments, $(\tilde{\mu}_t^n)_{t\geq 0}$ converges in $\mathcal{M}_0^+$ uniformly in time to $(\tilde{\mu}_t)_{t\geq 0}$ solution to (4.25), and since $\tilde{\mu}_t^n \geq 0$ for all $n$, we deduce $\tilde{\mu}_t \geq 0$ for all $t \geq 0$. The uniqueness for (4.25) follows from similar computations.

Thus, by Lemma 4.10. we deduce existence and uniqueness of $(\mu_t)_{t\geq 0}$ solution to (4.22), and since $\tilde{\mu}_t \geq 0$ we have $\mu_t \geq 0$ for all $t \geq 0$.

Finally, it remains to prove (4.23). For this, we apply (4.22) with $\phi(x) \equiv 1$, remark that $\left(\tilde{A}1\right)(x,y) \leq 0$ and that $(L1)(x) \leq \overline{F}(k-1)\beta(\Theta)$. Since $\mu_t \geq 0$ for all $t \geq 0$, this implies

$$\|\mu_t\|_{VT} = \int_0^\infty \mu_t(dx) \leq \|\mu_0\|_{VT} + \overline{F}(k-1)\beta(d\Theta)\int_0^t \|\mu_s\|_{VT}\, ds.$$

Using the Gronwall Lemma, we conclude

$$\sup_{[0,t]} \|\mu_s\|_{VT} \leq \left\|\mu^{in}\right\|_{VT} e^{Ct} \quad \text{for all} \quad t \geq 0,$$

where $C$ is a positive constant depending only on $\overline{K}$, $\overline{F}$ and $\beta$. We point out that the term $\sup_{[0,t]} \|\nu_s\|_{VT}$ is not involved since it is relied to the coagulation part of the equation, which is negative and bounded by 0. This ends the proof of Lemma 4.8. □

*Proof of Proposition 4.6.* We define the following implicit scheme for (4.21): $\mu_t^0 = \mu^{in}$ for all $t \geq 0$ and for $n \geq 0$,

(4.29)

$$\begin{cases} \dfrac{d}{dt}\displaystyle\int_0^\infty \phi(x)\mu_t^{n+1}(dx) & = \displaystyle\int_0^\infty\int_0^\infty (\tilde{A}\phi)(x,y)\mu_t^{n+1}(dx)\mu_t^n(dy) + \int_0^\infty (L\phi)(x)\,\mu_t^{n+1}(dx) \\ \mu_0^{n+1} & = \mu^{in}. \end{cases}$$

First, from Lemma 4.8. for $n \geq 0$ we have existence of $(\mu_t^{n+1})_{t\geq 0}$ unique and non-negative solution to (4.29) whenever $(\mu_t^n)_{t\geq 0}$ is non-negative and $\sup_{[0,t]} \|\mu_s^n\|_{VT} < \infty$ for all $t \geq 0$. Hence, since $\mu^{in} \in \mathcal{M}_0^+$, by recurrence we deduce existence, uniqueness and non-negativity of $(\mu_t^{n+1})_{t\geq 0}$ for all $n \geq 0$ solution to (4.29).

Moreover, from (4.23), this solution is bounded uniformly in $n$ on $[0,t]$ for all $t \geq 0$ since this bound does not depend on $\mu_t^n$, i.e.,

$$\sup_{n\geq 1}\sup_{[0,t]} \|\mu_s^{n+1}\|_{VT} \leq C_t \,\|\mu^{in}\|_{VT}. \tag{4.30}$$

Next, note that the operators $\tilde{A}$ and $L$ are bounded:

$$\|L\phi\|_\infty \leq \overline{F}(k+1)\beta(\Theta)\|\phi\|_\infty, \tag{4.31}$$

$$\left\|\int_0^\infty (\tilde{A}\phi)(\,\cdot\,,y)\mu(dy)\right\|_\infty \leq \frac{3}{2}\overline{K}\|\phi\|_\infty\,\|\mu\|_{VT}. \tag{4.32}$$

From (4.32) and (4.31),

$$\begin{aligned} &\frac{d}{dt}\int_0^\infty \phi(x)\left(\mu_t^{n+1}(dx) - \mu_t^n(dx)\right) \\ &\quad = \int_0^\infty\int_0^\infty (\tilde{A}\phi)(x,y)\left(\mu_t^{n+1}(dx)\mu_t^n(dy) - \mu_t^n(dx)\mu_t^{n-1}(dy)\right) \\ &\quad\quad + \int_0^\infty (L\phi)(x)\,\left(\mu_t^{n+1} - \mu_t^n\right)(dx) \\ &\quad = \int_0^\infty\int_0^\infty (\tilde{A}\phi)(x,y)\left[\left(\mu_t^{n+1} - \mu_t^n\right)(dx)\mu_t^n(dy) + \mu_t^n(dx)\left(\mu_t^n - \mu_t^{n-1}\right)(dy)\right] \\ &\quad\quad + \int_0^\infty (L\phi)(x)\,\left(\mu_t^{n+1} - \mu_t^n\right)(dx) \\ &\quad \leq \frac{3}{2}\overline{K}\|\phi\|_\infty\,\|\mu_t^n\|_{VT}\left[\int_0^\infty \left|\mu_t^{n+1} - \mu_t^n\right|(dx) + \int_0^\infty \left|\mu_t^n - \mu_t^{n-1}\right|(dy)\right] \\ &\quad\quad + \overline{F}(k+1)\beta(\Theta)\|\phi\|_\infty\left\|\mu_t^{n+1} - \mu_t^n\right\|_{VT}, \end{aligned}$$

implying,

$$\begin{aligned}\frac{d}{dt}\int_0^\infty \phi(x)\left(\mu_t^{n+1}(dx)-\mu_t^n(dx)\right) &\leq \|\phi\|_\infty\left(\frac{3}{2}\overline{K}\,\|\mu_t^n\|_{VT}+\overline{F}(k+1)\beta(\Theta)\right)\left\|\mu_t^{n+1}-\mu_t^n\right\|_{VT}\\ &\quad+\frac{3}{2}\overline{K}\,\|\phi\|_\infty\,\|\mu_t^n\|_{VT}\,\left\|\mu_t^n-\mu_t^{n-1}\right\|_{VT}.\end{aligned}$$

We integrate on $t$, take the sup over $\|\phi\|_\infty \leq 1$, and use (4.30), to deduce that there exist two constants $C_{1,t}$ and $C_{2,t}$ depending on $t$ but not on $n$ such that

$$\left\|\mu_t^{n+1}-\mu_t^n\right\|_{VT} \leq C_{1,t}\int_0^t \left\|\mu_s^{n+1}-\mu_s^n\right\|_{VT}ds + C_{2,t}\int_0^t \left\|\mu_s^n-\mu_s^{n-1}\right\|_{VT}ds.$$

Note that the difference of initial conditions vanishes since they are the same. We obtain using the Gronwall Lemma.

$$\left\|\mu_t^{n+1}-\mu_t^n\right\|_{VT} \leq C_{2,t}\,e^{t\,C_{1,t}}\int_0^t \left\|\mu_s^n-\mu_s^{n-1}\right\|_{VT}ds.$$

Hence, by usual arguments, $(\mu_t^n)_{t\geq 0}$ converges in $\mathcal{M}_0^+$ uniformly in time to the desired solution, which is also unique. Moreover, for some finite constant $C$ depending on $t$, $\overline{K}$, $\overline{F}$ and $\beta$, this solution satisfies (4.18) by (4.30).

This concludes the proof of Proposition 4.6. □

**Existence and uniqueness for $c^{in} \in \mathcal{M}_\lambda^+ \cap \mathcal{M}_2^+$.-**

We are no longer under (4.17), more generally we assume Hypotheses 2.1. and 2.2. This paragraph is devoted to show existence in the case where the initial condition satisfies:

$$c^{in} \in \mathcal{M}_\lambda^+ \cap \mathcal{M}_2^+.$$

First, for $n \geq 1$, we consider $c^{in,n}(dx) = \mathbb{1}_{[1/n,n]}c^{in}(dx)$, this measure belongs to $\mathcal{M}_0^+$ and satisfies

$$\sup_{n\geq 1} M_\lambda(c^{in,n}) \leq M_\lambda(c^{in}). \tag{4.33}$$

We also note that $\left(F^{c^{in,n}}\right)$ converges towards $F^{c^{in}}$ in $L^1(0,\infty;x^{\lambda-1}\,dx)$ as $n\to\infty$. Define $K_n$ by $K_n(x,y)=K(x,y)\wedge n$ for $(x,y)\in(0,\infty)^2$. Notice that (2.2) and (2.3) warrant that

$$\begin{aligned} K_n(x,y) &\leq \kappa_0(x+y)^\lambda,\\ (x^\lambda\wedge y^\lambda)|\partial_x K_n(x,y)| &\leq \kappa_1 x^{\lambda-1}y^\lambda.\end{aligned} \tag{4.34}$$

Furthermore, we consider the set $\Theta(n)$ defined by $\Theta(n)=\left\{\theta\in\Theta:\theta_1\leq 1-\frac{1}{n}\right\}$, we consider also the projector

$$\begin{aligned}\psi_n:\ \Theta &\to \Theta_n\\ \theta &\mapsto \psi_n(\theta)=(\theta_1,\ldots,\theta_n,0,\ldots),\end{aligned} \tag{4.35}$$

and we put

$$\beta_n = \mathbb{1}_{\theta\in\Theta(n)}\beta\circ\psi_n^{-1}. \tag{4.36}$$

The measure $\beta_n$ can be seen as the restriction of $\beta$ to the projection of $\Theta(n)$ onto $\Theta_n$. Note that $\Theta(n)\subset\Theta(n+1)$ and that since we have excluded the degenerated cases $\theta_1=1$ we have $\bigcup_n \Theta(n)=\Theta$.

Then, $K_n$, $F$ and $\beta_n$ satisfy (4.17) (use (4.10)) and since $c^{in,n}\in\mathcal{M}_0^+$, we have from Proposition 4.6. (recall Remark 4.7.) that for each $n\geq 1$, there exists a $(c^{in,n},K_n,F,\beta_n,\lambda)$-weak solution $(c_t^n)_{t\geq 0}$ to (2.12).

Note that since we have fragmentation it is not evident that $M_\lambda(c_t)$ remains finite in time. We need to control $M_\lambda(c_t)$ to verify (2.11). For this, we set $\phi(x) = x^\lambda$, from (2.12) and since $(A\phi)(x,y) \le 0$ we have

$$\begin{aligned} \frac{d}{dt}\int_0^\infty x^\lambda c_t^n(dx) &= \frac{1}{2}\int_0^\infty\int_0^\infty K_n(x,y)(A\phi)(x,y)c_t^n(dx)\,c_t^n(dy) \\ &\quad + \int_\Theta\int_0^\infty F(x)\left(\sum_{i\ge 1}\theta_i^\lambda - 1\right) x^\lambda c_t^n(dx)\beta_n(d\theta) \\ &\le \kappa_2\, C_\beta^\lambda M_\lambda(c_t^n), \end{aligned}$$

where we used that clearly $C_{\beta_n}^\lambda \le C_\beta^\lambda$ for all $n \ge 1$ (recall (2.7)). Note also that if $\sum_{i\ge1}\theta_i^\lambda - 1 < 0$ then $M_\lambda(c_t^n) < M_\lambda(c_0)$.

Using the Gronwall Lemma and (4.33) we deduce, for all $t \ge 0$

$$\sup_{n\ge1}\sup_{[0,t]} M_\lambda(c_s^n) \le C_t, \tag{4.37}$$

where $C_t$ is a positive constant. Next, apply (2.12) with $\phi(x) = x^2$ and since $\sum_{i\ge1}\theta_i^2 - 1 \le 0$ the fragmentation part is negative. In [9, Lemma A.3. *(ii)*] was shown that there exists a constant $C$ depending only on $\lambda$ and $\kappa_0$ such that $K_n(x,y)|(A\phi)(x,y)| \le K(x,y)|(A\phi)(x,y)| \le C(x^2y^\lambda + x^\lambda y^2)$. Thus,

$$\begin{aligned} \frac{d}{dt}\int_0^\infty x^2 c_t^n(dx) &\le \frac{C}{2}\int_0^\infty\int_0^\infty (x^2y^\lambda + x^\lambda y^2)\,c_t^n(dx)\,c_t^n(dy) \\ &= C M_\lambda(c_t^n) M_2(c_t^n). \end{aligned}$$

Using the Gronwall Lemma, we obtain

$$M_2(c_t^n) \le M_2(c^{in})\, e^{C\int_0^t M_\lambda(c_s^n)ds},$$

for $t \ge 0$ and for each $n \ge 1$. We point out that $x^2 \notin \mathcal{H}_\lambda$, but we can proceed as in Remark 4.7, considering $\phi(x) = x^2 \wedge A$ with $A > 0$ and making $A$ tend to infinity.

Hence, using (4.37) we get

$$\sup_{n\ge1}\sup_{[0,t]} M_2(c_s^n) \le C_t, \tag{4.38}$$

where $C_t$ is a positive constant.

We set $E_n(t,x) = F^{c_t^{n+1}}(x) - F^{c_t^n}(x)$ and define $R_n(t,x) = \int_0^x z^{\lambda-1} sign(E_n(t,x))dz$. Recall (4.6) and (4.7),

$$
\begin{aligned}
&\frac{d}{dt}\int_0^\infty x^{\lambda-1}|E_n(t,x)|dx \\
&= \frac{1}{2}\int_0^\infty \partial_x R_n(t,z)\int_0^z\int_0^z \mathbb{1}_{[z,\infty)}(x+y)K_{n+1}(x,y)(c_t^{n+1}(dy)\,c_t^{n+1}(dx) - c_t^n(dy)\,c_t^n(dx))\,dz \\
&\quad -\frac{1}{2}\int_0^\infty \partial_x R_n(t,z)\int_z^\infty\int_z^\infty K_{n+1}(x,y)(c_t^{n+1}(dy)\,c_t^{n+1}(dx) - c_t^n(dy)\,c_t^n(dx))\,dz \\
&\quad +\int_0^\infty \partial_x R_n(t,z)\int_\Theta \sum_{i\geq 1}\int_{z/\theta_i}^\infty F(x)(c_t^{n+1} - c_t^n)(dx)\beta_{n+1}(d\theta)\,dz \\
&\quad -\int_0^\infty \partial_x R_n(t,z)\int_\Theta \int_z^\infty F(x)(c_t^{n+1} - c_t^n)(dx)\beta_{n+1}(d\theta)\,dz \\
&\qquad +\frac{1}{2}\int_0^\infty \partial_x R_n(t,z)\int_0^z\int_0^z \mathbb{1}_{[z,\infty)}(x+y)\,(K_{n+1}(x,y) - K_n(x,y))\,c_t^n(dy)\,c_t^n(dx)\,dz \\
&\qquad -\frac{1}{2}\int_0^\infty \partial_x R_n(t,z)\int_z^\infty\int_z^\infty (K_{n+1}(x,y) - K_n(x,y))\,c_t^n(dy)\,c_t^n(dx)\,dz \\
&\qquad +\int_0^\infty \partial_x R_n(t,z)\int_\Theta \sum_{i\geq 1}\int_{z/\theta_i}^\infty F(x)\,c_t^n(dx)(\beta_{n+1} - \beta_n)(d\theta)\,dz \\
&\qquad -\int_0^\infty \partial_x R_n(t,z)\int_\Theta \int_z^\infty F(x)\,c_t^n(dx)(\beta_{n+1} - \beta_n)(d\theta)\,dz.
\end{aligned}
$$

Thus, after some computations, we obtain

$$
\frac{d}{dt}\int_0^\infty x^{\lambda-1}|E_n(t,x)|dx = I_1^n(t,x) + I_2^n(t,x) + I_3^n(t,x) + I_4^n(t,x), \tag{4.39}
$$

where $I_1^n(t,x)$ and $I_2^n(t,x)$ are respectively the equivalent terms to the coagulation and fragmentation parts in (4.8) and

$$
\begin{aligned}
I_3^n(t,x) &= \frac{1}{2}\int_0^\infty\int_0^\infty (K_{n+1}(x,y) - K_n(x,y))\,(AR_n(t))(x,y)c_t^n(dy)\,c_t^n(dx) \\
I_4^n(t,x) &= \int_0^\infty F(x)\int_\Theta (BR_n(t))(\theta,x)(\beta_{n+1} - \beta_n)(d\theta)c_t^n(dx),
\end{aligned}
$$

which are the terms resulting of the approximation.

Exactly as in (4.3), since the bounds in (4.34) do not depend on $n$ and that $\beta_n$ satisfies (2.7) uniformly in $n$, we get

$$
I_1^n(t,x) + I_2^n(t,x) \leq C_1 M_\lambda(c_t^n + c_t^{n+1})\int_0^\infty x^{\lambda-1}|E_n(t,x)|\,dx + C_2\int_0^\infty x^{\lambda-1}|E_n(t,x)|\,dx. \tag{4.40}
$$

Next, since

$$
K_{n+1}(x,y) - K_n(x,y) = \mathbb{1}_{\{K(x,y)>n+1\}} + (K(x,y)-n)\mathbb{1}_{\{n<K(x,y)\leq n+1\}} \leq \mathbb{1}_{\{K(x,y)>n\}} \leq \frac{K(x,y)^2}{n^2}
$$

and using (4.5), we have

$$\begin{aligned} |I_3^n(t,x)| &= \frac{1}{2}\left|\int_0^\infty\int_0^\infty (K_{n+1}(x,y)-K_n(x,y))\,(AR_n(t))(x,y)c_t^n(dy)\,c_t^n(dx)\right| \\ &\le \frac{1}{2}\int_0^\infty\int_0^\infty \frac{K(x,y)^2}{n^2}\,|(AR_n(t))(x,y)|\,c_t^n(dy)\,c_t^n(dx) \\ &\le \frac{2^{2\lambda+1}\kappa_0^2}{2\lambda n^2}\int_0^\infty\int_0^\infty (x\vee y)^{2\lambda}(x\wedge y)^\lambda c_t^n(dy)\,c_t^n(dx) \\ &\le \frac{C}{n^2}M_{2\lambda}(c_t^n)M_\lambda(c_t^n) \;\le\; \frac{1}{n^2}\,C_t, \end{aligned} \tag{4.41}$$

we have used $M_{2\lambda}(c_t)\le M_\lambda(c_t)+M_2(c_t)$ together with (4.37) and (4.38).

Finally, since $\int_\Theta (BR_n(t))(\theta,x)\beta_n(d\theta) = \int_\Theta (BR_n(t))(\psi_n(\theta),x)\mathbb{1}_{\{\theta\in\Theta(n)\}}\beta(d\theta)$, we have

$$\begin{aligned} &|I_4^n(t,x)| \\ &= \left|\int_0^\infty F(x)\int_\Theta \Big\{ [(BR_n(t))(\psi_{n+1}(\theta),x)-(BR_n(t))(\psi_n(\theta),x)]\,\mathbb{1}_{\Theta(n)\cap\Theta(n+1)} \right. \\ &\qquad\left. +\,(BR_n(t))(\psi_{n+1}(\theta),x)\mathbb{1}_{\Theta(n+1)\setminus\Theta(n)}\Big\}\beta(d\theta)c_t^n(dx)\right| \\ &\le \int_0^\infty F(x)\int_\Theta |R_n(t,\theta_{n+1}x)|\,\mathbb{1}_{\Theta(n+1)\cap\Theta(n)}\beta(d\theta)c_t^n(dx) \\ &\qquad + \int_0^\infty F(x)\int_\Theta \left|\sum_{i=1}^{n+1} R_n(t,\theta_i x)-R_n(t,x)\right|\mathbb{1}_{\Theta(n+1)\setminus\Theta(n)}\beta(d\theta)c_t^n(dx) \\ &\le C\int_0^\infty x^\lambda c_t^n(dx)\int_\Theta \theta_{n+1}^\lambda\mathbb{1}_{\{\Theta(n+1)\cap\Theta(n)\}}\beta(d\theta) \\ &\quad + C\int_0^\infty x^\lambda c_t^n(dx)\int_\Theta\left[\sum_{i\ge 2}\theta_i^\lambda+(1-\theta_1)^\lambda\right]\mathbb{1}_{\{\Theta(n+1)\setminus\Theta(n)\}}\beta(d\theta) \\ &\le C_t\int_\Theta \theta_{n+1}^\lambda\beta(d\theta) + C_t\int_\Theta\left[\sum_{i\ge 2}\theta_i^\lambda+(1-\theta_1)^\lambda\right]\mathbb{1}_{\{\Theta(n+1)\setminus\Theta(n)\}}\beta(d\theta), \end{aligned} \tag{4.42}$$

we used (4.37). Gathering (4.40), (4.41) and (4.42) in (4.39) and noting $C(\theta):=\sum_{i\ge 2}\theta_i^\lambda+(1-\theta_1)^\lambda$, we obtain

$$\begin{aligned} \frac{d}{dt}\int_0^\infty x^{\lambda-1}|E_n(t,x)|dx &\le C_tM_\lambda(c^{in})\int_0^\infty x^{\lambda-1}|E_n(t,x)|\,dx+\frac{1}{n^2}\,C_t \\ &\quad +C_t\int_\Theta \theta_{n+1}^\lambda\beta(d\theta)+C_t\int_\Theta C(\theta)\mathbb{1}_{\{\Theta(n+1)\setminus\Theta(n)\}}\beta(d\theta). \end{aligned}$$

Thus by the Gronwall Lemma we obtain

$$\begin{aligned} \int_0^\infty x^{\lambda-1}\left|F^{c_t^{n+1}}(x)-F^{c_t^n}(x)\right|dx &\le C_t\bigg(\int_0^\infty x^{\lambda-1}\left|F^{c^{in,n+1}}(x)-F^{c^{in,n}}(x)\right|dx+\frac{1}{n^2} \\ &\quad +\int_\Theta \theta_{n+1}^\lambda\beta(d\theta)+\int_\Theta C(\theta)\mathbb{1}_{\{\Theta(n+1)\setminus\Theta(n)\}}\beta(d\theta)\bigg), \end{aligned}$$

for $t \geq 0$ and $n \geq 1$ and where $C_t$ is a positive constant depending on $\lambda$, $\kappa_0$, $\kappa_1$, $\kappa_2$, $\kappa_3$, $C_\beta^\lambda$, $t$ and $c^{in}$. Recalling that

$$t \mapsto F^{c_t^n} \text{ belongs to } \mathcal{C}\left([0,\infty); L^1(0,\infty; x^{\lambda-1}dx)\right),$$

for each $n \geq 1$ by Lemma 4.2. and Lemma 4.5, and since the last three terms in the right-hand side of the inequality above are the terms of convergent series, we conclude that $\left(t \mapsto F^{c_t^n}\right)_{n\geq 1}$ is a Cauchy sequence in $\mathcal{C}\left([0,\infty); L^1(0,\infty; x^{\lambda-1}dx)\right)$ and there is

$$f \in \mathcal{C}\left([0,\infty); L^1(0,\infty; x^{\lambda-1}dx)\right)$$

such that

$$\lim_{n\to\infty} \sup_{s\in[0,t]} \int_0^\infty x^{\lambda-1} \left|F^{c_s^{n+1}}(x) - f(s,x)\right| dx = 0 \quad \text{for each } \; t \in [0,\infty). \tag{4.43}$$

As a first consequence of (4.43), we obtain that $x \mapsto f(t,x)$ is a non-deacreasing and non-negative function for each $t \in [0,\infty)$. Furthermore,

$$\lim_{\varepsilon\to 0} \sup_{s\in[0,t]} \left[\int_0^\varepsilon x^{\lambda-1} f(s,x)dx + \int_{1/\varepsilon}^\infty x^{\lambda-1} f(s,x)dx\right] = 0 \tag{4.44}$$

for each $t \in (0,\infty)$ since $f \in \mathcal{C}\left([0,\infty); L^1(0,\infty; x^{\lambda-1}dx)\right)$.

We will show that this convergence implies tightness of $(c_t^n)_{n\geq 1}$ in $\mathcal{M}_\lambda^+$, uniformly with respect to $s \in [0,t]$. We consider $\varepsilon \in (0,1/4)$, and since $x \mapsto F^{c_s^n}(x)$ is non-decreasing and $\lambda \in (0,1]$, it follows from Lemma 4.2.:

$$\int_0^\varepsilon x^\lambda c_t^n(dx) + \int_{1/\varepsilon}^\infty x^\lambda c_t^n(dx) \leq \int_0^\varepsilon x^{\lambda-1} F^{c_t^n}(x)dx + \int_{1/(2\varepsilon)}^\infty x^{\lambda-1} F^{c_t^n}(x)dx.$$

The Lebesgue dominated convergence Theorem, (4.43) and (4.44) give

$$\lim_{\varepsilon\to 0} \sup_{n\geq 1} \sup_{s\in[0,t]} \left[\int_0^\varepsilon x^\lambda c_t^n(dx) + \int_{1/\varepsilon}^\infty x^\lambda c_t^n(dx)\right] = 0, \tag{4.45}$$

for every $t \in [0,\infty)$. Denoting by $c_t(dx) := -\partial_x f(t,x)$ the derivative with respect to $x$ of $f$ in the sense of distributions for $t \in (0,\infty)$, we deduce from (4.37), (4.43) and (4.45) that $c_t(dx) \in \mathcal{M}_\lambda^+$ with $M_\lambda(c_t) \leq e^{\kappa_2 C_\beta^\lambda t} M_\lambda(c^{in})$.

Consider now $\phi \in \mathcal{C}_c^1((0,\infty))$ and recall that $|\phi'(x)| \leq C x^{\lambda-1}$ for some positive constant $C$. On the one hand, the time continuity of $f$ implies that

$$t \mapsto \int_0^\infty \phi(x) c_t(dx) = \int_0^\infty \phi'(x) f(t,x)dx$$

is continuous on $[0,\infty)$. On the other hand, the convergence (4.43) entails

$$\begin{aligned}
\lim_{n\to\infty} \sup_{s\in[0,t]} \left|\int_0^\infty \phi(x)(c_s^n - c_s)(dx)\right| &= \lim_{n\to\infty} \sup_{s\in[0,t]} \left|\int_0^\infty \phi'(x)\left(F^{c_s^n}(x) - F^{c_s}(x)\right)dx\right| \\
&\leq \lim_{n\to\infty} \sup_{s\in[0,t]} \left|C\int_0^\infty x^{\lambda-1}\left(F^{c_s^n}(x) - F^{c_s}(x)\right)dx\right| \\
&= 0,
\end{aligned} \tag{4.46}$$

for every $t \geq 0$. We then infer from (4.45), (4.46), Lemma 4.1., (4.4) and a density argument that for every $\phi \in \mathcal{H}_\lambda$, the map $t \mapsto \int_0^\infty \phi(x) c_t(dx)$ is continuous and

$$\lim_{n\to\infty} \sup_{s\in[0,t]} \left| \int_0^\infty \phi(x)(c_s^n - c_s)(dx) \right| = 0,$$

$$\lim_{n\to\infty} \sup_{s\in[0,t]} \left| \frac{1}{2} \int_0^\infty \int_0^\infty (A\phi)(x,y)] K(x,y)(c_s^n(dx)c_s^n(dy) - c_s(dx)c_s(dy)) + \int_0^\infty F(x) \int_\Theta (B\phi)(\theta, x)\beta(d\theta)(c_s^n - c_s)(dx) \right| = 0.$$

We may thus pass to the limit as $n \to \infty$ in the integrated form of (2.12) for $(c_t^n)_{t\geq 0}$ and deduce that for all $t \geq 0$ and $\phi \in \mathcal{H}_\lambda$, we have

$$\begin{aligned}
\int_0^\infty \phi(x) c_t(x)\, dx = & \int_0^\infty \phi(x) c^{in}(x)\, dx \\
& + \frac{1}{2} \int_0^\infty \int_0^\infty [\phi(x+y) - \phi(x) - \phi(y)] K(x,y) c_t(dx) c_t(dy) \\
& + \int_0^\infty \int_\Theta \left[ \sum_{i=1}^\infty \phi(\theta_i x) - \phi(x) \right] F(x)\beta(d\theta) c_t(dx).
\end{aligned} \tag{4.47}$$

Classical arguments then allows us to differentiate (4.47) with respect to time and conclude that $(c_t^n)_{t\geq 0}$ is a $(c^{in}, K, F, \beta, \lambda)$-weak solution to (1.1).

**Existence and uniqueness for $c^{in} \in \mathcal{M}_\lambda^+$.-**

We have shown existence for $c^{in} \in \mathcal{M}_\lambda^+ \cap \mathcal{M}_2^+$. Now we are going to extend the previous result to an initial condition only in $\mathcal{M}_\lambda^+$. For this, we consider $(a_n)_{n\geq 1}$ and $(A_n)_{n\geq 1}$ two sequences in $\mathbb{R}^+$ such that $a_n$ is non-increasing and converging to 0 and $A_n$ non-decreasing and tending to $+\infty$ with $0 < a_0 \leq A_0$. We set $B_n = [a_n, A_n]$ and define

$$c^{in,n}(dx) := c^{in}|_{B_n}(dx),$$

note that trivially we have $M_2(c^{in,n}) < \infty$. Next, we call $(\tilde{c}_t^n)_{t\geq 1}$ the $(c^{in,n}, K, F, \beta, \lambda)$-weak solution to (1.1) constructed in the previous section.

Owing to Proposition 4.3. and (4.3), we have for $t \geq 0$ and $n \geq 1$

$$\int_0^\infty x^{\lambda-1} \left| F^{\tilde{c}_t^{n+1}}(x) - F^{\tilde{c}_t^n}(x) \right| dx \leq e^{C_t} \int_0^\infty x^{\lambda-1} \left| F^{c^{in,n+1}}(x) - F^{c^{in,n}}(x) \right| dx,$$

Next, we have

$$\begin{aligned}
& \int_0^\infty x^{\lambda-1} \left| F^{c^{in,n+1}}(x) - F^{c^{in,n}}(x) \right| dx \\
&= \int_0^{+\infty} x^{\lambda-1} \left| \int_0^{+\infty} \mathbb{1}_{[x,+\infty)}(y) \left( c^{in}|_{B_n} - c^{in}|_{B_{n+1}} \right)(dy) \right| dx \\
&= \int_0^{+\infty} x^{\lambda-1} \int_0^{+\infty} \mathbb{1}_{[x,+\infty)}(y) \left( \mathbb{1}_{[a_{n+1},a_n)}(y) + \mathbb{1}_{[A_n,A_{n+1})}(y) \right) c^{in}(dy) dx,
\end{aligned}$$

note that since $\sum_{n\geq 0} \left[ \mathbb{1}_{[a_{n+1},a_n)}(y) + \mathbb{1}_{[A_n,A_{n+1})}(y) \right] \leq \mathbb{1}_{\mathbb{R}^+}(y)$ the term in the right-hand of the last inequality is summable. We conclude that $\left(t \mapsto F^{\tilde{c}_t^n}\right)_{n\geq 1}$ is a Cauchy sequence in $\mathcal{C}\left([0,\infty); L^1(0,\infty; x^{\lambda-1}dx)\right)$

and there is

$$f \in \mathcal{C}\left([0,\infty); L^1(0,\infty; x^{\lambda-1}dx)\right),$$

such that

$$\lim_{n\to\infty} \sup_{s\in[0,t]} \int_0^\infty x^{\lambda-1} \left| F^{\tilde{c}_s^{n+1}}(x) - f(s,x) \right| dx = 0 \quad \text{for each} \quad t \in [0,\infty).$$

and we conclude using the same arguments as in the previous case, setting $c_t := -\partial_x f(t,x)$ in the sense of distributions, that $(c_t)_{t\geq 0}$ is a $(c^{in}, K, F, \beta, \lambda)$-weak solution to (1.1) in the sense of Definition 2.4.

This completes the proof of Theorem 2.5. □

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