

ON PERMANENT AND BREAKING WAVES IN HYPERELASTIC RODS AND RINGS

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ABSTRACT. We prove that the only global strong solution of the periodic rod equation vanishing in at least one point $(t_0, x_0) \in \mathbb{R}^+ \times \mathbb{S}^1$ is the identically zero solution. Such conclusion holds provided the physical parameter γ of the model (related to the finger deformation tensor) is outside some neighborhood of the origin and applies in particular for the Camassa–Holm equation, corresponding to $\gamma = 1$. We also establish the analogue of this unique continuation result in the case of non-periodic solutions defined on the whole real line with vanishing boundary conditions at infinity. Our analysis relies on the application of new local-in-space blowup criteria and involves the computation of several best constants in convolution estimates and weighted Poincaré inequalities.

1. INTRODUCTION

Motivations. This paper is devoted to the study of periodic solutions of the rod equation. A first physical motivation comes from the study of the response of hyper-elastic rings under the action of an initial radial stretch. As the nonlinear dispersive waves propagating inside it could eventually lead to cracks, an important problem is the determination of conditions that must be fulfilled in order to prevent their formation. A main issue of the present paper will be a precise description of crack mechanisms inside such rings.

A second reason for studying periodic solutions is that periodic waves spontaneously arise also in hyper-elastic rods: indeed, it has been recently observed that the solitary waves propagating inside an ideally infinite length rod can suddenly feature a transition into waves with finite period as their amplitude increases, see [18].

Our third motivation comes from the the study of waves inside channels in the shallow water regime. If the motion of small amplitude waves is usually modeled by the KdV equation, larger amplitude waves, and in particular breaking waves, are more accurately described by the Camassa–Holm equation. Indeed the Camassa–Holm equation better captures the genuinely nonlinear behavior of such waves and, contrary to KdV, admits both permanent solutions and solutions that blow up in finite time. See [7, 13, 15]. In fact, the Camassa–Holm equation is a particular case (corresponding to the case $\gamma = 1$) of the rod equation below.

Let $\mathbb{S} = \mathbb{R}/\mathbb{Z}$ be the unit circle. The Cauchy problem for the periodic rod equation is written as follows:

$$(1.1) \quad \begin{cases} u_t + \gamma u u_x = -\partial_x p * \left(\frac{3-\gamma}{2} u^2 + \frac{\gamma}{2} u_x^2 \right), & t \in (0, T), x \in \mathbb{S}, \\ u(0, x) = u_0(x). \end{cases}$$

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The real parameter γ is related to the Finger deformation tensor of the material. Both positive and negative values γ are admissible. We refer to [18] for more details on the physical background and the mathematical derivation of the model.

The function p is kernel of the convolution operator $(1 - \partial_x^2)^{-1}$. Its expression is

$$(1.2) \quad p(x) = \frac{(x - [x] - 1/2)}{2 \sinh(1/2)},$$

where $[\cdot]$ denotes the integer part. Thus, the kernel p is a continuous 1-periodic function.

The Camassa–Holm case $\gamma = 1$ is somewhat particular, as in this case (1.1) inherits a bi-Hamiltonian structure and the equation is completely integrable. For this reason, many important results valid for the Camassa–Holm equation do not go through in the general case. For example, in the case of the Camassa–Holm equation on the real line, a striking necessary and sufficient condition for the global existence of strong solution can be given in terms on the initial potential $y_0 = u_0 - u_{0,xx}$, see [25]. On the other hand, very little is known on the global existence of strong solutions when $\gamma \neq 1$: it can even happen (when $\gamma = 3$) that all nonzero solutions blow up in finite time. Smooth solitary waves that are global strong solutions were constructed for $0 < \gamma < 1$, see [18]. These are essentially the only known examples of global smooth solutions.

Our working assumption will be that u_0 belongs to the Sobolev space $H^s(\mathbb{S})$, for some $s > 3/2$. Then, for any $\gamma \in \mathbb{R}$, the Cauchy problem for the rod equation is locally well-posed, in the sense that there exists a maximal time $0 < T^* \leq \infty$ and a unique solution $u \in C([0, T^*), H^s(\mathbb{S})) \cap C^1([0, T^*), H^{s-1}(\mathbb{S}))$. Moreover, the solution u depends continuously on the initial data. It is also known that u admits several invariant integrals, among which the energy integral,

$$E(u) = \int_{\mathbb{S}} (u^2 + u_x^2) dx.$$

In particular, because of the conservation of the Sobolev H^1 -norm, the solution $u(x, t)$ remains uniformly bounded up to the time T^* . On the other hand, if $T^* < \infty$ then $\limsup_{t \rightarrow T} \|u(t)\|_{H^s} = \infty$ ($s > 3/2$) and the precise blowup scenario, often named wave breaking mechanism, is the following:

$$(1.3) \quad T^* < \infty \iff \liminf_{t \rightarrow T^*} \left(\inf_{x \in \mathbb{S}} \gamma u_x(t, x) \right) = -\infty,$$

See, *e.g.* [16].

Quick overview of the main results. We will state our two main theorems in the next section after preparing some notations. Loosely, the first theorem asserts that if $|\gamma|$ is *not too small*, then there exist a constant $\beta_\gamma > 0$ such that if

$$(1.4) \quad u'_0(x_0) > \beta_\gamma |u_0(x_0)| \quad \text{if } \gamma < 0, \quad \text{or} \quad u'_0(x_0) < -\beta_\gamma |u_0(x_0)| \quad \text{if } \gamma > 0$$

in one point $x_0 \in \mathbb{S}$, then the solution arising from $u_0 \in H^s(\mathbb{S})$ must blow up in finite time.

Our second theorem quantifies the above result: it precises what “ $|\gamma|$ not too small” means, addressing also the delicate issue of finding sharp estimates for β_γ . For example, in the particular case of the periodic *Camassa–Holm* equation we get that a sufficient condition for the blowup is:

$$(1.5) \quad \exists x_0 \in \mathbb{S}: \quad u'_0(x_0) < -\sqrt{\frac{5}{2} - \frac{3}{2} \cdot \frac{\cosh \frac{1}{2} \cosh \frac{3}{2} - 1}{\sinh \frac{1}{2} \sinh \frac{3}{2}}} |u_0(x_0)|.$$

An analogue but weaker result was recently established in our previous paper [3], where we dealt with non-periodic solutions on the whole real line with vanishing boundary conditions as $x \rightarrow \infty$. In the present paper, we take advantage of the specific structure of periodic solutions in order to make improvements on our previous work in two directions. First of all in the case of the circle we find better coefficients: as a comparison, the coefficient on the right-hand side of (1.5) is approximatively equal to -0.515 ; on the other hand the blowup criterion for the non-periodic Camassa–Holm equation (with vanishing boundary condition at infinity) found in [3] was $u'_0(x_0) < -|u_0(x_0)|$, for some $x_0 \in \mathbb{R}$, the coefficient -1 being optimal.

More importantly, the analogue blowup result for the rod equation on \mathbb{R} could be established in [3] only in the range $1 \leq \gamma \leq 4$ in the non-periodic case. But the relevant estimates on the circle that we will establish turn out to be much stronger. They allow us to cover blowup results of periodic solutions *e.g.* for arbitrary large γ (or γ negative and arbitrary small). The following corollary of Theorem 2.1 is therefore specific to periodic solutions:

Corollary 1.1. *There exists an absolute constant β_∞ (it can be checked numerically that $\beta_\infty = 0.295\dots$) with the following property. If $u_0 \in H^s(\mathbb{S})$, with $s > 3/2$, is such that for some $x_0 \in \mathbb{S}$, $u'_0(x_0) > \beta_\infty|u_0(x_0)|$, or otherwise $u'_0(x_0) < -\beta_\infty|u_0(x_0)|$, then the solutions (unique, but depending on γ) of the rod equation (1.1) arising from u_0 blow up in finite time respectively if $\gamma \ll -1$ or $\gamma \gg 1$. In both cases, the maximal existence time is $T^* = O(\frac{1}{|\gamma|})$ as $\gamma \rightarrow \infty$.*

The most important feature of our blowup criteria (1.4)-(1.5) is that they are *local-in-space*. This means that these criteria involve a condition *only on a small neighborhood of a single point* of the datum. Their validity is somewhat surprising, as equation (1.1) is non-local. A huge number of previous papers addressed the blowup issue of solutions to equation (1.1), (see, *e.g.* [6, 7, 11–13, 29–33] (the older references only dealt with the Camassa–Holm equation). But the corresponding blowup criteria systematically involved the computation of some global quantities of u_0 : typically, conditions of the form $u'_0(x_0) < -c_\gamma \|u_0\|_{H^1(\mathbb{S}^1)}$ or some other other integral conditions on u_0 , or otherwise antisymmetry conditions, etc.

The main idea (inspired to us combining those of [3, 11, 24]) will be to study the evolution of $u + \beta u_x$ and $u - \beta u_x$ along the trajectories of the flow map of γu , for an appropriate parameter $\beta = \beta(\gamma)$ to be determined in order to optimize the blowup result. The main technical issue of the present paper will be the study of a two-parameters family of minimization problems. Such minimization problems arise when computing the best constants in the relevant convolution estimates. Calculus of variation tools have been already successfully used for the study of blowup criteria of the rod equation, see, *e.g.*, [29–33]. The difference with respect to these papers is that our approach requires the minimization of non-coercive functionals. Once such minima are computed, the proof of the blowup can follow the steps of [3].

We finish this introduction by mentioning a second simple consequence of our main Theorem 2.1, that seems to be of independent interest. Its result applies in particular to the case $\gamma = 1$. It is a new result (at best of our knowledge) also for the Camassa–Holm equation.

Corollary 1.2. *Let $u \in C([0, \infty[, H^s(\mathbb{S})) \cap C^1([0, \infty[, H^{s-1}(\mathbb{S}))$, be a global solution of the rod equation (1.1) with $\gamma \leq \gamma_1^-$ or $\gamma \geq \gamma_1^+$ (where $\gamma_1^- = -1.036\dots$ and $\gamma_1^+ = 0.269\dots$). If u vanishes at some point $(t_0, x_0) \in [0, \infty) \times \mathbb{S}^1$, then u must be the trivial solution: $u(x, t) \equiv 0$ for all $t \geq 0$ and $x \in \mathbb{S}$.*

Corollary 1.2 improves an earlier result by A. Constantin and J. Escher, [14] asserting that the trivial solution is the only global solution of the periodic Camassa–Holm equation such that *for all* $t \geq 0$, $\exists x_t \in \mathbb{S}$ such that $u(t, x_t) = 0$. Basically, we manage to replace their condition “ $\forall t \dots$ ” by the much weaker one “ $\exists t \dots$ ”. More importantly, unlike [14], our approach will make use of only few properties of the equation, and is more suitable for generalizations.

It might be surprising that our results *a priori* excludes a small neighborhood of the origin for the parameter γ . In fact, such restriction might be purely technical. However, one should observe that $\gamma = 0$ *must be excluded*. The reason is that for $\gamma = 0$ all solutions arising from $u_0 \in H^s(\mathbb{S})$ are global in time. Indeed, the blowup scenario (1.3) is never fulfilled. It is worth noticing that for $\gamma = 0$ the rod equation reduces to the BBM equation introduced by Benjamin, Bona and Mahony in [1] as a model for surface wave in channels.

We will establish a result similar to that of Corollary 1.2 for non-periodic solutions vanishing for $x \rightarrow \infty$ (see Corollary 7.1 at the end of the paper for the precise statement). In this case, we will need that that $u(t_0, x_1) = u(t_0, x_2) = 0$ in at least two different points, or otherwise that $u(t_0, \cdot)$ decays sufficiently fast as $x \rightarrow \infty$ to conclude that $u(t, x) \equiv 0$ for all $(t, x) \in \mathbb{R}^+ \times \mathbb{R}$.

Organization of the paper. In Section 2 we state our main results, Theorem 2.1 and Theorem 2.2. The first theorem is proved in Section 3. In Section 4 we study a minimization problem that will play an important role for the proof of Theorem 2.2, given in Section 5. Corollary 1.1 will immediately follows from assertion (ii) of this theorem. We will study in more detail the Camassa–Holm equation in Section 6. Corollary 1.2 is established in Section 7 where we also provide some new results for non-periodic solutions. An appendix showing the agreement between our theorems and some numerical results concludes the paper.

2. THE MAIN RESULTS

We start preparing some notations. For any real constant α and β , let $I(\alpha, \beta) \geq -\infty$ defined by

$$(2.1) \quad I(\alpha, \beta) = \inf \left\{ \int_0^1 (p + \beta p_x) (\alpha u^2 + u_x^2) dx : u \in H^1(0, 1), u(0) = u(1) = 1 \right\}.$$

Proposition 3.3 below will characterize the set of parameters (α, β) for which the functional appearing in (2.1) is bounded from below as well as the subset for which the infimum is achieved.

We also introduce, for $\gamma \in \mathbb{R}^*$, the quantity $\beta_\gamma \in [0, +\infty]$ defined by

$$(2.2) \quad \beta_\gamma = \inf \left\{ \beta \in \mathbb{R}^+ : \beta^2 + I\left(\frac{3-\gamma}{\gamma}, \beta\right) - \frac{3-\gamma}{\gamma} \geq 0 \right\},$$

with the usual convention that $\beta_\gamma = +\infty$ if the infimum is taken on the empty set.

Our main result is the following blowup theorem:

Theorem 2.1. *Let $\gamma \in \mathbb{R}^*$ be such that $\beta_\gamma < +\infty$. Let $u_0 \in H^s(\mathbb{S})$ with $s > 3/2$ and assume that there exists $x_0 \in \mathbb{S}$, such that*

$$(2.3) \quad u'_0(x_0) > \beta_\gamma |u_0(x_0)| \quad \text{if } \gamma < 0, \quad \text{or} \quad u'_0(x_0) < -\beta_\gamma |u_0(x_0)| \quad \text{if } \gamma > 0.$$

Then the corresponding solution u of equation (1.1) in $C([0, T^*), H^s(\mathbb{S})) \cap C^1([0, T^*), H^{s-1}(\mathbb{S}))$ arising from u_0 blows up in finite time. Moreover, the maximal time T^* is estimated by

$$(2.4) \quad T^* \leq \frac{2}{\gamma \sqrt{u_0'(x_0)^2 - \beta_\gamma^2 u_0(x_0)^2}}.$$

Theorem 2.1 is meaningful only if γ is such that $\beta_\gamma < +\infty$. The validity of such condition is *a priori* not clear, as it might happen that the set in equation (2.2) is empty. An important part of the present paper will be devoted to the technical issue of discussing the validity of the condition $\beta_\gamma < +\infty$ and next estimating β_γ . This will be done by establishing some sharp estimates on $I(\alpha, \beta)$.

In order to state our next theorem, let us introduce the complex number

$$\mu = \frac{1}{2} \sqrt{1 + 4(3 - \gamma)/\gamma}, \quad \gamma \neq 0,$$

where $\sqrt{1 + 4(3 - \gamma)/\gamma}$ denotes any of the two complex square roots. We also consider the four constants (that will be constructed in (5.8) below):

$$\begin{aligned} \gamma_1^- &= -1.036\dots & \gamma_1^+ &= 0.269\dots \\ \gamma_2^- &= -1.508\dots & \gamma_2^+ &= 0.575\dots \end{aligned}$$

Theorem 2.2.

- (i) For any $\gamma \in (-\infty, \gamma_1^-] \cup [\gamma_1^+, +\infty)$, we have $\beta_\gamma < +\infty$, so that Theorem 2.1 applies in such range. More precisely, if $\gamma \in (-\infty, \gamma_2^-] \cup [\gamma_2^+, \infty)$, then

$$(2.5) \quad \beta_\gamma \leq \sqrt{\frac{3}{\gamma} - \frac{1}{2} - \mu \cdot \frac{\cosh \frac{1}{2} \cosh \mu - 1}{\sinh \frac{1}{2} \sinh \mu}}.$$

If otherwise $\gamma \in [\gamma_2^-, \gamma_1^-] \cup [\gamma_1^+, \gamma_2^+]$, then β_γ can be estimated as in inequality (5.13) below (in (5.13), P_ν denotes Legendre function of the first kind of degree ν).

- (ii) The limit $\beta_\infty = \lim_{\gamma \rightarrow \pm\infty} \beta_\gamma$ does exist and

$$(2.6) \quad \beta_\infty \leq \sqrt{\frac{\sqrt{3}(1 - \cosh \frac{1}{2} \cos \frac{\sqrt{3}}{2})}{2 \sinh \frac{1}{2} \sin \frac{\sqrt{3}}{2}} - \frac{1}{2}} = 0.296\dots$$

It is worth observing that we have $\beta_\gamma = 0$ if and only if $\gamma = 3$. This can be checked directly from the definitions (2.1)-(2.2). Accordingly, the right-hand side in (2.5) vanishes for $\gamma = 3$. In particular, we recover the known fact (see [16]) that for $\gamma = 3$ any nonzero initial datum $u_0 \in H^s(\mathbb{S})$ (with $s > 3/2$) gives rise to a solution that blows up in finite time. Moreover, the maximal time T^* satisfies

$$T^* \leq \frac{2}{3} \sqrt{-\inf_{x \in \mathbb{S}} u_0'(x)}.$$

The upper bounds (2.5)–(2.6) are not optimal but, strikingly, they are *almost sharp*. Indeed we can compute the numerical approximation of β_γ with arbitrary high precision. We find that the error between the above bounds and the numerical value is only of order 10^{-3} . For example, we find $\beta_\infty = 0.295\dots$ that is indeed very close to the bound (2.6). Moreover, in the Camassa–Holm case, we find numerically that $\beta_1 = 0.513\dots$. This is in good agreement with estimate (2.5), that for $\gamma = 1$ provided us with the bound $\beta_1 \leq 0.515\dots$

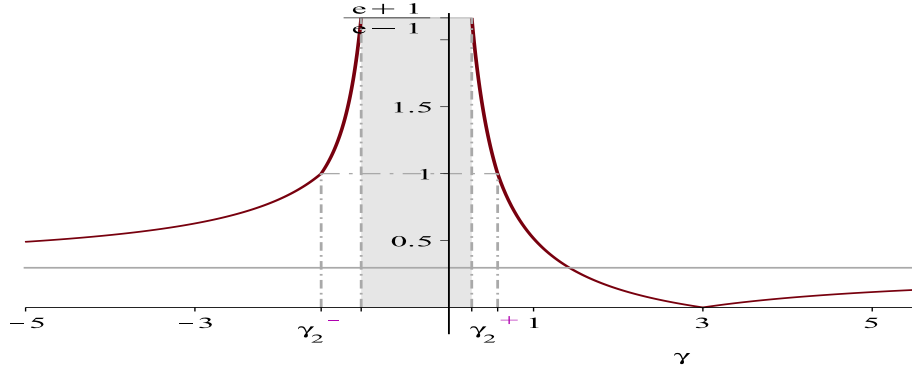


FIGURE 1. The upper-bound estimate of β_γ given by Theorem 2.2. The estimate is valid outside the interval $[\gamma_1^-, \gamma_1^+]$ (gray region).

We will devote the appendix to a more detailed discussion of the numerical results. Such analysis will also show that the effective range of applicability of Theorem 2.1 is slightly larger than the range $\gamma \in (-\infty, \gamma_1^-] \cup [\gamma_1^+, \infty)$ predicted by Theorem 2.2.

This paper does not address the problem of the continuation of the solution after the blowup, but this issue is extensively studied in the literature. See, *e.g.* [4, 5, 21].

3. FIRST PROPERTIES OF $I(\alpha, \beta)$ AND PROOF OF THEOREM 2.1

For any real β , let us consider the 1-periodic function

$$(3.1) \quad \omega(x) = p(x) + \beta p'(x),$$

where p is the kernel introduced in (1.2) and p' denotes the distributional derivative on \mathbb{R} , that agrees in this case with the classical a.e. pointwise derivative on $\mathbb{R} \setminus \mathbb{Z}$.

We would like to make use of ω as a weight function. The non-negativity condition $\omega \geq 0$ is equivalent to the inequality $\cosh(1/2) \geq \pm \beta \sinh(1/2)$, *i.e.*, to the condition

$$(3.2) \quad -\frac{e+1}{e-1} \leq \beta \leq \frac{e+1}{e-1}.$$

Throughout this section, we will work under the above condition on β .

Let us introduce the weighted Sobolev space

$$(3.3) \quad E_\beta = \left\{ u \in L^1_{\text{loc}}(0, 1) : \|u\|_{E_\beta}^2 \equiv \int_0^1 \omega(x)(u^2 + u_x^2)(x) \, dx < \infty \right\},$$

where the derivative is understood in the distributional sense. Notice that E_β agrees with the classical Sobolev space $H^1(0, 1)$ when $|\beta| < \frac{e+1}{e-1}$, as in this case ω is bounded and bounded away from 0, and the two norms $\|\cdot\|_{E_\beta}$ and $\|\cdot\|_{H^1}$ are equivalent. The situation is different for $\beta \pm \frac{e+1}{e-1}$ as E_β is strictly larger than $H^1(0, 1)$ in this case. Indeed, we have

$$(3.4) \quad \omega(x) = \frac{2e}{(e-1)^2} \sinh(x), \quad x \in (0, 1), \quad (\text{if } \beta = \frac{e+1}{e-1}).$$

The elements of $E_{(e+1)/(e-1)}$, after modification on a set of measure zero, are thus continuous on $(0, 1]$, but may be unbounded for $x \rightarrow 0^+$ (for instance, $|\log(x/2)|^{1/3} \in E_{(e+1)/(e-1)}$). In the same way,

$$(3.5) \quad \omega(x) = \frac{2e}{(e-1)^2} \sinh(1-x), \quad x \in (0, 1), \quad (\text{if } \beta = -\frac{e+1}{e-1}).$$

After modification on a set of measure zero, the elements of $E_{-(e+1)/(e-1)}$ are continuous on $[0, 1)$, but may be unbounded for $x \rightarrow 1^-$.

Let us consider the closed subspace $E_{\beta,0}$ of E_β defined (with slightly abusive notation) as follows :

$$(3.6) \quad \begin{aligned} E_{\beta,0} &= H_0^1(0,1) = \{u \in H^1(0,1) : u(0) = u(1) = 0\}, & \text{if } |\beta| < \frac{e+1}{e-1}, \\ E_{\beta,0} &= \{u \in E_\beta : u(1) = 0\}, & \text{if } \beta = \frac{e+1}{e-1}, \\ E_{\beta,0} &= \{u \in E_\beta : u(0) = 0\}, & \text{if } \beta = -\frac{e+1}{e-1}. \end{aligned}$$

Equivalently, $E_{\beta,0}$ could be defined as the closure of $C_c^\infty(0,1)$ in E_β . This is of course classical for $|\beta| < \frac{e+1}{e-1}$. For $\beta = \pm \frac{e+1}{e-1}$ our claim follows from the next lemma.

Lemma 3.1. *Let $\beta = \pm \frac{e+1}{e-1}$ and $u \in E_\beta$. Then there exists a sequence $(u_n) \subset C_c^\infty(0,1)$ such that $\|u_n - u\|_{E_\beta} \rightarrow 0$.*

Proof. We consider only the case $\beta = \frac{e+1}{e-1}$ as in the other case the proof is similar. By the density of $C_c^\infty(0,1)$ in H_0^1 , we just have to approximate u by elements in H_0^1 . Consider the sequence $(u_n) \subset H_0^1$ defined by

$$u_n(x) = \begin{cases} u(x) & \text{if } x \in (\frac{1}{n}, 1), \\ \log(1/2)u(\frac{1}{n})\log(nx/2))^{-1} & \text{if } x \in (0, \frac{1}{n}). \end{cases}$$

To see that $\|u_n - u\|_{E_\beta} \rightarrow 0$, first observe that $u(x) = O(|\log(x)|^{1/2})$ as $x \rightarrow 0^+$. Indeed,

$$\begin{aligned} |u(x)| &= \left| \int_x^1 u'(y) dy \right| \leq \left(\int_0^1 \omega(y)u'(y)^2 dy \right)^{1/2} \left(\int_x^1 \frac{1}{\omega(y)} dy \right)^{1/2} \\ &\leq C\|u\|_{E_\beta} \sqrt{|\log x|}. \end{aligned}$$

by Cauchy-Schwarz inequality, and because $\omega(x) \sim cx$ as $x \rightarrow 0$. Using $u(\frac{1}{n}) = O(\sqrt{\log n})$, it is an elementary computation to prove that

$$\int_0^{1/n} \omega(x)(u_n(x)^2 + u_n'(x)^2) \rightarrow 0.$$

This in turn implies $u_n \rightarrow u$ in the E_β norm and our claim follows. \square

The elements of $E_{\beta,0}$ satisfy to the weighted Poincaré inequality below:

Lemma 3.2. *For all $-\frac{e+1}{e-1} \leq \beta \leq \frac{e+1}{e-1}$, there exists a constant $C > 0$ such that*

$$(3.7) \quad \forall v \in E_{\beta,0} : \int_0^1 \omega(x)v(x)^2 dx \leq C \int_0^1 \omega(x)v_x(x)^2 dx.$$

Proof. The validity of such inequality is obvious for $|\beta| < \frac{e+1}{e-1}$. Indeed, in this case there exist two constants c_1 and c_2 such that on the interval $(0, 1)$ we have $0 < c_1 \leq \omega(x) \leq c_2$ and the validity of (3.7) is reduced to that of the classical Poincaré inequality without weight.

In the limit case $\beta = \frac{e+1}{e-1}$, we can observe that, from (3.4), the only zero of the function $\omega(x) = \frac{2e}{(e-1)^2} \sinh(x)$ in the closure of $(0, 1)$ is of order one. Then the weight $\omega(x)$ satisfies the necessary and sufficient condition for the weighted Poincaré inequality to hold, see [28]. For reader's convenience we prove directly inequality (3.7) exhibiting an explicit constant. Recall that $v(x) = O(|\log(x)|^{1/2})$ as $x \rightarrow 0^+$. In particular, for all $v \in E_{(e+1)/(e-1), 0}$, we have $[\omega(x)v(x)^2]_{0+}^{1-} = 0$. Then integrating by parts and next using the Cauchy-Schwarz inequality we obtain, for $v \in E_{(e+1)/(e-1), 0}$:

$$\frac{1}{4} \left(\int_0^1 \cosh v^2 \right)^2 = \left(\int_0^1 \sinh v v_x \right)^2 \leq \left(\int_0^1 \sinh v^2 \right) \left(\int_0^1 \sinh v_x^2 \right).$$

Next observe that

$$\forall x \in (0, 1), \quad \cosh(x) \geq \frac{e^2 + 1}{e^2 - 1} \sinh(x).$$

Combining these two estimates, we obtain

$$(3.8) \quad \int_0^1 \sinh(x) v(x)^2 dx \leq \frac{4(e^2 - 1)^2}{(e^2 + 1)^2} \int_0^1 \sinh(x) v_x(x)^2 dx.$$

Then (3.7) holds, *e.g.*, with $C = \frac{4(e^2 - 1)^2}{(e^2 + 1)^2}$. In the other limit case $\beta = -\frac{e+1}{e-1}$, $\omega(x) = \frac{2e}{(e-1)^2} \sinh(1-x)$. Therefore, we can reduce to the previous case with a change of variables. \square

The constant (3.8) is far from being optimal. In fact, we will find the best constant in Remark 4.1, as a byproduct of the analysis performed Section 4.

This being observed, let us go back to our minimization problem

$$(3.9) \quad I(\alpha, \beta) = \inf \left\{ \int_0^1 (p + \beta p') (\alpha u^2 + u_x^2) dx : u \in H^1(0, 1), u(0) = u(1) = 1 \right\}.$$

Proposition 3.3. *We have*

$$I(\alpha, \beta) > -\infty \iff \begin{cases} -\frac{e+1}{e-1} \leq \beta \leq \frac{e+1}{e-1}, \\ \alpha > -1/C(\beta), \end{cases}$$

where $C(\beta) > 0$ is the best constant in the weighted Poincaré inequality (3.7). Moreover, if $|\beta| < \frac{e+1}{e-1}$, then $I(\alpha, \beta)$ is in fact a minimum and there is only one minimizer $u \in H^1(0, 1)$ with $u(0) = u(1) = 1$.

Proof. Putting $u = v + 1$ and observing that $\int_0^1 \omega(x) dx = 1$, we see that

$$(3.10) \quad I(\alpha, \beta) = \alpha + \inf \{ J(v) : v \in H_0^1(0, 1) \},$$

where

$$(3.11) \quad J(v) = \int_0^1 \omega(x) (\alpha v^2 + v_x^2 + 2\alpha v)(x) dx.$$

Assume that $I(\alpha, \beta) > -\infty$. Then $|\beta| \leq \frac{e+1}{e-1}$, otherwise we would get a contradiction by taking a sequence of the form $\chi(x) \cos(nx)$ with χ smooth and such that $\text{supp}(\chi) \subset \text{supp}(\omega^-) \cap (0, 1)$,

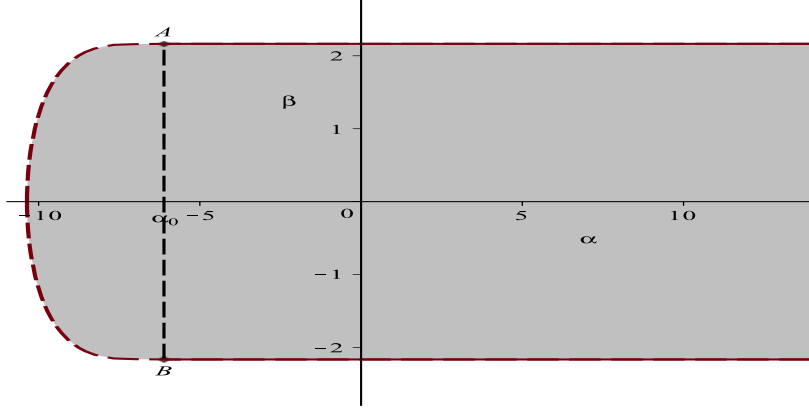


FIGURE 2. The set of points (α, β) such that $I(\alpha, \beta) > -\infty$ (gray region). The infimum is achieved in H_0^1 on its interior and in the larger space $E_{\beta,0}$ on the upper and lower boundaries $\beta = \pm \frac{e+1}{e-1}$. The abscissa $\alpha_0 = -1/C(\mp \frac{e+1}{e-1})$ of the vertical dashed segment is the largest zero of a suitable Legendre function (see Subsection 4.2).

where ω^- denotes the negative part of ω . To prove the second inequality $\alpha > -1/C(\beta)$, we only have to treat the case $\alpha < 0$. Applying the inequality

$$\int_0^1 \omega(\alpha n^2 v^2 + n^2 v_x^2 + 2\alpha n v) \geq I(\alpha, \beta) - \alpha$$

valid for all $v \in H_0^1(0, 1)$ and all $n \in \mathbb{N}$ and letting $n \rightarrow \infty$, we get

$$\int_0^1 \omega v^2 \leq -\frac{1}{\alpha} \int_0^1 \omega v_x^2.$$

Then we get $\alpha \geq -1/C(\beta)$. But in fact the inequality is strict, as otherwise we could take a sequence (v_n) such that $(-\alpha \int_0^1 \omega v_n^2) / (\int \omega v_{x,n}^2) \rightarrow 1$ and $\int \alpha \omega v_n \rightarrow -\infty$ to get a contradiction.

Conversely, assume that $|\beta| \leq \frac{e+1}{e-1}$. By the weighted Poincaré inequality (3.7), the map $v \mapsto \int_0^1 \omega v_x^2$ defines on $E_{\beta,0}$ an equivalent norm. As $\alpha < -1/C(\beta)$, the symmetric bilinear form $B(v_1, v_2) = \int_0^1 \omega(\alpha v_1 v_2 + v_1' v_2')$ is coercive on the Hilbert space $E_{\beta,0}$. Applying the Lax-Milgram theorem yields the existence and the uniqueness of a minimizer $\bar{v} \in E_{\beta,0}$ for the functional J . But $H_0^1(0, 1) \subset E_{\beta,0}$, so in particular, we get $I(\alpha, \beta) > -\infty$. Moreover, if $|\beta| < \frac{e+1}{e-1}$, then recalling $E_{\beta,0} = H_0^1$ we see that $I(\alpha, \beta)$ is in fact a minimum, achieved at $\bar{u} = 1 + \bar{v} \in H^1$. \square

The next proposition provides some useful information on $I(\alpha, \beta)$.

Proposition 3.4. *The function $(\alpha, \beta) \mapsto I(\alpha, \beta) \in \mathbb{R} \cup \{-\infty\}$, defined for all $(\alpha, \beta) \in \mathbb{R}^2$, is concave with respect to each one of its variables and is even with respect to the variable β . Moreover,*

$$(3.12) \quad \forall \alpha, \beta \in \mathbb{R}, \quad -\infty \leq I(\alpha, \frac{e+1}{e-1}) \leq I(\alpha, \beta) \leq I(\alpha, 0) \leq \alpha.$$

Proof. The concavity property follows from the fact that $I(\alpha, \beta)$ is defined as an infimum of affine functions of the variables α and β . To prove that $I(\alpha, \beta) = I(\alpha, -\beta)$, we can observe that

$$(p + \beta p')(x) = (p - \beta p')(1 - x)$$

and conclude making the change of variable $y = 1 - x$ inside the integral in (3.9).

To prove the last inequality in (3.12), consider the constant function $u \equiv 1$ and observe that $I(\alpha, 0) \leq \alpha \int_0^1 (p + \beta p')(x) dx = \alpha$. The other inequalities follow from the concavity and parity properties of the map $\beta \mapsto I(\alpha, \beta)$. The result of Proposition 3.3 is also needed for the second inequality. \square

Next lemma motivates the introduction of the quantity $I(\alpha, \beta)$ in relation with the rod equation.

Lemma 3.5. *For any $\alpha, \beta \in \mathbb{R}$ and all $u \in H^1(\mathbb{S})$ the following convolution estimate holds:*

$$(3.13) \quad \forall x \in \mathbb{S}, \quad (p + \beta p') * (\alpha u^2 + u_x^2)(x) \geq I(\alpha, \beta) u(x)^2$$

and $I(\alpha, \beta)$ is the best possible constant.

Remark 3.6. This lemma is interesting when $I(\alpha, \beta) > -\infty$, but its statement is true also when $I(\alpha, \beta) = -\infty$. In this case the sentence “ $-\infty$ is the best possible constant” should be understood as a negative result: the convolution estimate (3.13) breaks down whenever (α, β) does not fulfill the conditions of Proposition 3.3.

Proof. Let $\delta = \delta(\alpha, \beta)$ be some (possibly negative) constant. Because of the invariance under translations, we have that

$$(3.14) \quad (p + \beta p') * (\alpha u^2 + u_x^2) \geq \delta u^2$$

holds true for all $u \in H^1(\mathbb{S})$ if and only if

$$(p + \beta p') * (\alpha u^2 + u_x^2)(1) \geq \delta u^2(1)$$

holds true for all $u \in H^1(\mathbb{S})$. But on the interval $(0, 1)$, $(p + \beta p')(1 - x) = (p - \beta p')(x)$. Hence,

$$(p + \beta p') * (\alpha u^2 + u_x^2)(1) = \int_0^1 (p - \beta p')(x)(\alpha u^2 + u_x^2)(x) dx.$$

Normalizing to obtain $u(1) = 1$ (and hence $u(0) = 1$ by the periodicity) we get that the best constant δ in inequality (3.14) satisfies $\delta = I(\alpha, -\beta) = I(\alpha, \beta)$. \square

Proof of Theorem 2.1. Let us first consider the case $u_0 \in H^3(\mathbb{S})$. We thus obtain a unique solution u of the rod equation (1.1), defined in some nontrivial interval $[0, T)$, and such that $u \in C([0, T), H^3(\mathbb{S})) \cap C^1((0, T), H^2(\mathbb{S}))$. As in [11], [24], the starting point is the analysis of the flow map $q(t, x)$, defined by

$$(3.15) \quad \begin{cases} q_t(t, x) = \gamma u(t, q(t, x)), & t \in (0, T), x \in \mathbb{R}, \\ q(0, x) = x. \end{cases}$$

As $u \in C^1([0, T), H^2)$, we see that u and u_x are continuous on $[0, T) \times \mathbb{R}$ and $x \mapsto u(t, x)$ is Lipschitz, uniformly with respect to t in any compact time interval in $[0, T)$. Then the flow map $q(t, x)$ is well defined by (3.15) in the whole time interval $[0, T)$ and $q \in C^1([0, T) \times \mathbb{R}, \mathbb{R})$.

From the rod equation

$$(3.16) \quad u_t + \gamma uu_x = -\partial_x p * \left(\frac{3-\gamma}{2} u^2 + \frac{\gamma}{2} u_x^2 \right),$$

differentiating with respect to the x variable and applying the identity $\partial_x^2 p * f = p * f - f$, we get

$$(3.17) \quad \begin{aligned} u_{tx} + \gamma uu_{xx} &= \frac{3-\gamma}{2} u^2 - \frac{\gamma}{2} u_x^2 - p * \left(\frac{3-\gamma}{2} u^2 + \frac{\gamma}{2} u_x^2 \right) \\ &= \frac{3}{2(\alpha+1)} [\alpha u^2 - u_x^2 - p * (\alpha u^2 + u_x^2)]. \end{aligned}$$

Here we set

$$(3.18) \quad \alpha = \frac{3-\gamma}{\gamma}.$$

Let us introduce the two C^1 -functions of the time variable, depending on β ,

$$f(t) = (-u_x + \beta u)(t, q(t, x_0)), \quad \text{and} \quad g(t) = -(u_x + \beta u)(t, q(t, x_0)).$$

Computing the time derivative using the definition of the flow q , next using equations (3.16)-(3.17), we get

$$(3.19) \quad \begin{aligned} \frac{df}{dt}(t) &= [(-u_{tx} - \gamma uu_{xx}) + \beta(u_t + \gamma uu_x)](t, q(t, x_0)) \\ &= \frac{3}{2(\alpha+1)} [-\alpha u^2 + u_x^2 + (p - \beta p') * (\alpha u^2 + u_x^2)](t, q(t, x_0)), \end{aligned}$$

and

$$(3.20) \quad \frac{dg}{dt}(t) = \frac{3}{2(\alpha+1)} [-\alpha u^2 + u_x^2 + (p + \beta p') * (\alpha u^2 + u_x^2)](t, q(t, x_0)).$$

Let us first consider the case $\gamma > 0$. Then $\alpha > -1$. From the definition of β_γ (2.2) and the condition $\beta_\gamma < \infty$, we deduce that there exist $\beta \geq 0$ such that

$$(3.21) \quad \beta^2 \geq \alpha - I(\alpha, \beta).$$

Applying the convolution estimate (3.13) and recalling $I(\alpha, -\beta) = I(\alpha, \beta)$, we get, for all $\beta \geq 0$ satisfying (3.21),

$$\begin{aligned} \frac{df}{dt}(t) &\geq \frac{3}{2(\alpha+1)} [u_x^2 - (\alpha - I(\alpha, -\beta))u^2](t, q(t, x_0)) \\ &\geq \frac{3}{2(\alpha+1)} [u_x^2 - \beta^2 u^2](t, q(t, x_0)) \\ &= \frac{3}{2(\alpha+1)} f(t)g(t). \end{aligned}$$

In the same way,

$$\begin{aligned} \frac{dg}{dt}(t) &\geq \frac{3}{2(\alpha+1)} [u_x^2 - (\alpha - I(\alpha, \beta))u^2](t, q(t, x_0)) \\ &\geq \frac{3}{2(\alpha+1)} [u_x^2 - \beta^2 u^2](t, q(t, x_0)) \\ &= \frac{3}{2(\alpha+1)} f(t)g(t). \end{aligned}$$

The assumption $u'_0(x_0) < -\beta_\gamma|u(x_0)|$ guarantees that we may choose β satisfying (3.21), with $\beta - \beta_\gamma > 0$ is small enough in a such way that $u'_0(x_0) < -\beta|u_0(x_0)|$ For such a choice of β we have

$$f(0) > 0 \quad \text{and} \quad g(0) > 0.$$

The blowup of u will rely on the following basic property:

Lemma 3.7. *Let $0 < T^* \leq \infty$ and $f, g \in C^1([0, T^*), \mathbb{R})$ be such that, for some constant $c > 0$ and all $t \in [0, T^*)$,*

$$(3.22) \quad \begin{aligned} \frac{df}{dt}(t) &\geq cf(t)g(t) \\ \frac{dg}{dt}(t) &\geq cf(t)g(t). \end{aligned}$$

If $f(0) > 0$ and $g(0) > 0$, then

$$T^* \leq \frac{1}{c\sqrt{f(0)g(0)}} < \infty.$$

Proof. Let

$$\tau = \inf\{t \in [0, T^*): f(t) = 0 \text{ or } g(t) = 0\}.$$

The positivity of $f(0)$ and $g(0)$ imply that $\tau > 0$. Observe that we cannot have $\tau < T^*$. Indeed, otherwise (exchanging if necessary f with g) we would get $f(\tau) = 0$ and $f, g \geq 0$ on $[0, \tau]$. The differential inequality on f imply that f is monotone increasing on $[0, \tau]$ leading to the contradiction $0 = f(\tau) \geq f(0) > 0$. Hence, f and g are both monotone strictly increasing and positive in the whole interval $[0, T^*)$.

Now take $h(t) = \sqrt{f(t)g(t)}$. Using again assumption (3.22), next the arithmetic-geometric mean inequality we find, on $[0, T^*)$,

$$\begin{aligned} \frac{dh}{dt} &\geq \frac{c}{2\sqrt{fg}} fg(f+g), \\ &\geq cfg = ch^2, \end{aligned}$$

with $h(0) = \sqrt{f(0)g(0)}$. Substituting $v = 1/h$, we find that $T^* \leq \frac{1}{c\sqrt{h(0)}}$. \square

Theorem 2.1 is now established in the case $\gamma > 0$. When $\gamma < 0$ the proof is the same, excepted for the fact that, as $\alpha < -1$, the inequalities must be reversed. \square

4. THE MINIMIZATION PROBLEM IN THE LIMIT CASE $\beta = \frac{e+1}{e-1}$ AND IN THE CASE $\beta = 1$

4.1. The limit case $\beta = \frac{e+1}{e-1}$.

We will start considering the limit case

$$\beta = \frac{e+1}{e-1}.$$

In this case, according to formula (3.1), the weight function becomes

$$(4.1) \quad \omega(x) = \frac{2e}{(e-1)^2} \sinh(x), \quad x \in (0, 1).$$

Because of Proposition 3.4, we are led to assume also

$$(4.2) \quad \alpha > -1/C\left(\frac{e+1}{e-1}\right).$$

We start observing that

$$I(\alpha, \frac{e+1}{e-1}) = \tilde{I}(\alpha, \frac{e+1}{e-1}),$$

where

$$\begin{aligned} \tilde{I}(\alpha, \frac{e+1}{e-1}) &\equiv \inf \left\{ \int_0^1 (p + \beta p') (\alpha u^2 + u_x^2) dx : u \in E_{(e+1)/(e-1)} \text{ and } u(1) = 1 \right\} \\ &= \alpha + \inf \left\{ J(v) : v \in E_{(e+1)/(e-1),0} \right\} \end{aligned}$$

and the functional J is given by (3.11). Indeed, the inequality $I(\alpha, \frac{e+1}{e-1}) \geq \tilde{I}(\alpha, \frac{e+1}{e-1})$ is obvious and the reverse inequality follows by the same approximation argument that we used in Lemma 3.1.

The unique minimizer $v \in E_{(e+1)/(e-1),0}$ of the functional J (whose existence was obtained in the proof of Proposition 3.3) satisfies the Euler–Lagrange equation complemented with the right-boundary condition

$$(4.3) \quad \begin{cases} (\omega v_x)_x - \alpha \omega v = \alpha \omega & \text{in } x \in (0, 1), \\ v(1) = 0. \end{cases}$$

Problem (4.3) is undetermined, but we will see that among its solutions only one belongs to $E_{(e+1)/(e-1),0}$.

4.2. The case $\beta = \frac{e+1}{e-1}$ and $\alpha > -1/C(\frac{e+1}{e-1})$.

The Euler–Lagrange equation (4.3) reads

$$(4.4) \quad \sinh(x)v_{xx}(x) + \cosh(x)v_x(x) - \alpha \sinh v(x) = \alpha \sinh(x), \quad \text{for } x \in (0, 1).$$

To find the general solution, consider the change of unknown $v(x) = f(y(x))$, with $y = \cosh(x)$. Then equation (4.4) can be rewritten in the y variable as

$$(4.5) \quad (1 - y^2)f_{yy} - 2yf_y(y) + \alpha f(y) = -\alpha, \quad \text{with } y \in (1, \cosh 1).$$

The constant function $y \mapsto -1$ is a particular solution. Substituting $\alpha = \nu(\nu + 1)$, we recognize the usual form of a non-homogeneous second order Legendre ODE. We thus set

$$(4.6) \quad \nu(\alpha) = -\frac{1}{2} + \frac{1}{2}\sqrt{1 + 4\alpha} \in \{z \in \mathbb{C} : \Im(z) \geq 0\},$$

where the complex square root is taken in $\{z \in \mathbb{C} : \Im(z) \geq 0\}$. The general solution of equation (4.5) is thus

$$(4.7) \quad f_{\lambda,\mu} = -1 + \lambda P_{\nu(\alpha)}(y) + \mu Q_{\nu(\alpha)}(y), \quad y \in (1, \cosh 1),$$

where P_ν and Q_ν are the two associate Legendre functions respectively of the first and of the second kind, of degree ν . We recall that Q_ν has a logarithmic singularity at 1^+ , whereas P_ν is bounded as $y \rightarrow 1^+$. Moreover, P_ν is a polynomial when ν is an integer. Notice that the function $P_{\nu(\alpha)}(\cosh)$ does belong to $E_{(e+1)/(e-1)}$, but $Q_{\nu(\alpha)}(\cosh)$ does not, because $Q_{\nu(\alpha)}(\cosh x) \neq O(\sqrt{|\log(x)|})$ as $x \rightarrow 0^+$.

Hence, the only solution $\bar{v} = v_{\lambda,\mu}$ of (4.4), such that $\bar{v} \in E_{(e+1)/(e-1),0}$ is obtained taking $\mu = 0$ and $\lambda = 1/P_{\nu(\alpha)}(1)$. Thus,

$$(4.8) \quad \bar{v}(x) = -1 + \frac{P_{\nu(\alpha)}(\cosh x)}{P_{\nu(\alpha)}(\cosh 1)}, \quad x \in (0, 1).$$

Expression (4.8) makes sense provided the denominator is nonzero. Therefore, we can recast condition (4.2) on α , guaranteeing that $I(\alpha, \frac{e+1}{e-1}) > -\infty$, in the following equivalent form:

$$\alpha > \alpha_0 \simeq -6.113,$$

where α_0 is the largest zero of the function $\alpha \mapsto P_{\nu(\alpha)}(\cosh 1)$ (see Figure 4.2). The representation of $P_{\nu}(z)$ as an hyper-geometric series, convergent for $|\frac{1}{2} - \frac{z}{2}| < 1$, is

$$\begin{aligned} P_{\nu}(z) &= {}_2F_1\left(-\nu, \nu + 1, \frac{1}{2} - \frac{z}{2}\right) \\ &= \sum_{k=0}^{+\infty} \frac{\Gamma(-\nu + k)\Gamma(\nu + 1 + k)}{(k!)^2\Gamma(-\nu)\Gamma(\nu + 1)} \left(\frac{1}{2} - \frac{z}{2}\right)^k. \end{aligned}$$

Such representation holds true for $z = \cosh 1$, but computing α_0 analytically seems to be difficult. On the other hand, α_0 can be estimated via Newton's method.

For $\alpha > \alpha_0 = -1/C(\frac{e+1}{e-1})$, using that \bar{v} solves (4.3), $\bar{v}_x(0^+) = 0$, and $\int_0^1 \omega = 1$, we have

$$\begin{aligned} (4.9) \quad I(\alpha, \frac{e+1}{e-1}) &= \alpha + J(\bar{v}) = \alpha + \int_0^1 \omega(\alpha\bar{v}^2 + \bar{v}_x^2 + 2\alpha\bar{v}) \\ &= \alpha + \int_0^1 [(\omega\bar{v}_x)_x\bar{v} + \omega\bar{v}_x^2 + \alpha\omega\bar{v}] \\ &= \alpha + [\omega\bar{v}_x\bar{v}]_{0^+}^{1^-} + \alpha \int_0^1 \omega\bar{v} \\ &= \int_0^1 (\omega\bar{v}_x)_x = (\omega\bar{v}_x)(1^-) \\ &= \frac{(e+1)^2}{2e} \frac{P'_{\nu(\alpha)}}{P_{\nu(\alpha)}}(\cosh 1). \end{aligned}$$

Remark 4.1. Incidentally, we proved that $c = -(1/\alpha_0) \simeq 0.164$ is the best constant in the weighted Poincaré inequality below (valid for all $v \in C^1([0, 1])$ such that $v(1) = 0$,

$$\forall v \in C^1([0, 1]): \int_0^1 \sinh(x)v(x)^2 dx \leq c \int_0^1 \sinh(x)v_x(x)^2 dx.$$

4.3. The case $\beta = 1$.

The computation of $I(\alpha, \beta)$ in the case $0 \leq \beta < \frac{e+1}{e-1}$ is different than that of the previous section, as standard variational methods apply in the usual Sobolev space $H_0^1(0, 1)$. Moreover, in the case $\beta = 1$, the associated Euler–Lagrange boundary value problem can be explicitly solved. According to Proposition 3.3, the computation below will be valid for $\alpha > -1/C(1)$. As a byproduct of our calculations, we will find the explicit expression

$$C(1) = 4/(1 + 4\pi^2)$$

for the weighted Poincaré inequality.

Recall that $I(\alpha, \beta) = \alpha + \inf\{J(v) : v \in H_0^1(0, 1)\}$, where $J(v)$ was introduced in (3.11) and the weight function

$$(4.10) \quad \omega(x) = p(x) + \beta p'(x) = \frac{(1 + \beta)e^x + (1 - \beta)e^{1-x}}{2(e - 1)}, \quad x \in (0, 1)$$

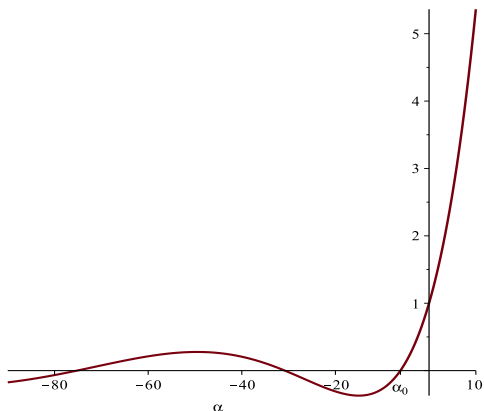


FIGURE 3. The function $\alpha \mapsto P_{\nu(\alpha)}(\cosh 1)$ and its largest zero α_0 .

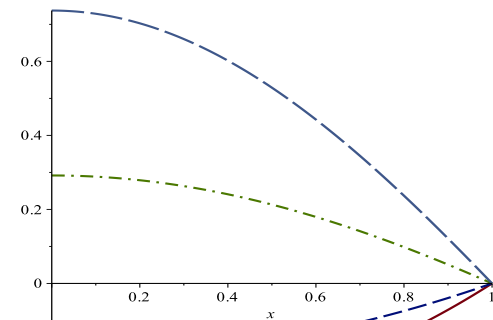


FIGURE 4. A few minimizers as given by formula (4.9), for $\alpha = 2, 1, -1, -2$ (from bottom to top).

is positive and bounded away from 0 on the interval $(0, 1)$. The unique minimizer $\bar{v} \in H_0^1$ is the solution of

$$(4.11) \quad \begin{cases} \omega v_{xx} + \omega_x v_x - \alpha \omega v = \alpha \omega & x \in (0, 1) \\ v(0) = v(1) = 0. \end{cases}$$

For $\beta = 1$, the weight ω reduces to $\omega(x) = e^x/(e - 1)$. Thus, the general solution of equation (4.11) for such choice of ω (at least for $\alpha \neq -1/4$) is

$$(4.12) \quad v(x) = -1 + \lambda e^{\nu(\alpha)x} + \mu e^{\overline{\nu(\alpha)}x}, \quad \lambda, \mu \in \mathbb{R},$$

where the complex number $\nu(\alpha)$ is understood as in (4.6). To simplify further the result let us set

$$(4.13) \quad \mu = \frac{1}{2} \sqrt{1 + 4\alpha} \in \{z \in \mathbb{C} : \Im(z) \geq 0\}.$$

Imposing the boundary conditions $v(0) = v(1) = 0$ we finally get the expression of minimizer of $I(\alpha, 1)$:

$$\bar{v}(x) = -1 + \frac{\sqrt{e} \sinh(\mu(\alpha)x) + \sinh(\mu(\alpha)(1 - x))}{e^{x/2} \sinh(\mu(\alpha))}.$$

The minimum $I(\alpha, 1)$ is thus given by

$$(4.14) \quad \begin{aligned} I(\alpha, 1) &= \alpha + J(\bar{v}) = \omega(1)\bar{v}_x(1) - \omega(0)\bar{v}_x(0) \\ &= -\frac{1}{2} + \mu \cdot \frac{\cosh \frac{1}{2} \cosh \mu - 1}{\sinh \frac{1}{2} \sinh \mu}. \end{aligned}$$

The above expression makes sense provided $\sinh(\mu(\alpha)) \neq 0$, *i.e.* for $\alpha \neq -\frac{1}{4} - k^2\pi^2$, with $k \in \mathbb{Z}$. But $\alpha = -1/4$ is in fact a removable singularity in (4.14). The restriction to be imposed on α is thus $\alpha > -\frac{1}{4} - \pi^2$.

5. PROOF OF THEOREM 2.2

We are now in the position of proving Theorem 2.2.

Proof of Theorem 2.2. Let us recall the definition of β_γ ,

$$(5.1) \quad \beta_\gamma = \inf \left\{ \beta \in \mathbb{R}^+ : \beta^2 + I(\alpha, \beta) - \alpha \geq 0 \right\},$$

where the one-to-one relation between α and γ is

$$(5.2) \quad \alpha = \frac{3 - \gamma}{\gamma}, \quad \text{or} \quad \gamma = \frac{3}{1 + \alpha}.$$

Using the results of the previous section we can now give explicit bounds from below for $I(\alpha, \beta)$ that can be used for the estimate of β_γ .

Using the concavity properties of the function $\beta \mapsto I(\alpha, \beta)$ we see that, for any fixed $\alpha > \alpha_0$, the infimum $I(\alpha, \beta)$ is bounded from below by piecewise affine function of the β variable. Namely,

$$(5.3) \quad \begin{cases} I(\alpha, \beta) \geq I(\alpha, 1), & \text{if } 0 \leq \beta \leq 1 \\ I(\alpha, \beta) \geq \frac{e-1}{2} \left(I(\alpha, \frac{e+1}{e-1}) - I(\alpha, 1) \right) \beta + \frac{e+1}{2} I(\alpha, 1) - \frac{e-1}{2} I(\alpha, \frac{e+1}{e-1}), & \text{if } 1 \leq \beta \leq \frac{e+1}{e-1}. \end{cases}$$

We denote by $R(\alpha, \beta)$, the function defined by the right-hand in (5.3), for $\alpha > \alpha_0$ and $0 \leq \beta \leq \frac{e+1}{e-1}$. The condition $\alpha > \alpha_0$ ensures that $I(\alpha, 1) \geq I(\alpha, \frac{e+1}{e-1}) > -\infty$, so that $R(\alpha, \beta)$ is finite.

The first issue is to find the condition on α guaranteeing $\beta_\gamma < +\infty$. Owing to the lower bound (5.3), a sufficient condition for this is that α is chosen in a such way that

$$(5.4) \quad \exists \beta \text{ such that } 0 \leq \beta \leq \frac{e+1}{e-1} : \quad \beta^2 + R(\alpha, \beta) - \alpha \geq 0$$

But condition (5.4) is equivalent to the following one:

$$(5.5) \quad 1 + I(\alpha, 1) - \alpha \geq 0 \quad \text{or} \quad \left(\frac{e+1}{e-1} \right)^2 + I(\alpha, \frac{e+1}{e-1}) - \alpha \geq 0.$$

Indeed, one implication is obvious and the converse one easily follows applying to the β -variable the elementary properties of quadratic polynomials. Let us make more explicit the last condition: because of the definition of $I(\alpha, \beta)$ and Proposition (3.4), the two functions $\alpha \mapsto I(\alpha, \frac{e+1}{e-1})$ and $\alpha \mapsto I(\alpha, 1)$ are both increasing, concave and vanishing at $\alpha = 0$. Therefore, there exists $\alpha_1^- < 0 < \alpha_1^+$ such that

$$(5.6) \quad \left(\frac{e+1}{e-1} \right)^2 + I(\alpha, \frac{e+1}{e-1}) - \alpha \geq 0 \iff \alpha_1^- \leq \alpha \leq \alpha_1^+.$$

For the same reason, there exists $\alpha_2^- < 0 < \alpha_2^+$ such that

$$(5.7) \quad 1 + I(\alpha, 1) - \alpha \geq 0 \iff \alpha_2^- \leq \alpha \leq \alpha_2^+.$$

The above zeros can be easily estimated via Newton's method. We find in this way

$$\alpha_1^- < \alpha_2^- < 0 < \alpha_2^+ < \alpha_1^+,$$

see also Figures 5-5.

According to (5.2), let us introduce the four constants

$$(5.8) \quad \begin{aligned} \gamma_1^- &= \frac{3}{1 + \alpha_1^-} = -1.036 \dots & \gamma_1^+ &= \frac{3}{1 + \alpha_1^+} = 0.269 \dots \\ \gamma_2^- &= \frac{3}{1 + \alpha_2^-} = -1.508 \dots & \gamma_2^+ &= \frac{3}{1 + \alpha_2^+} = 0.575 \dots \end{aligned}$$

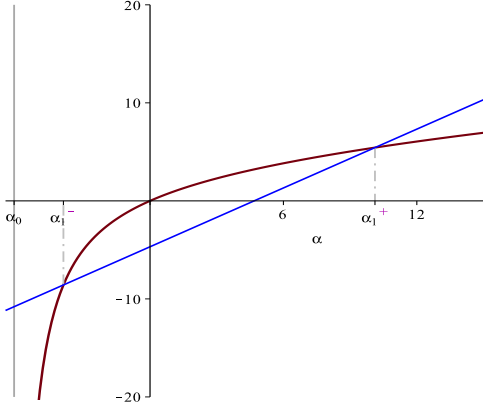


FIGURE 5. The plot of $\alpha \mapsto I(\alpha, \frac{e+1}{e-1})$ (see equation (4.9)) and of the straight line $\alpha \mapsto \alpha - (\frac{e+1}{e-1})^2$, intersecting the curve at α_1^- and α_1^+ .

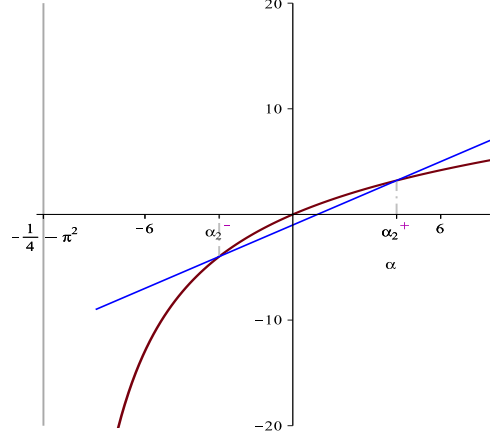


FIGURE 6. Plot of $\alpha \mapsto I(\alpha, 1)$ (see equation (4.14)) and of the straight line $\alpha \mapsto \alpha - 1$, intersecting the curve at α_2^- and α_2^+ .

In particular, for $\alpha_1^- \leq \alpha \leq \alpha_1^+$, we find that the set in (5.1) is nonempty: this means that in the range $\gamma \in (-\infty, \gamma_1^-] \cup [\gamma_1^+, +\infty)$ we get $\beta_\gamma < +\infty$, and Theorem 2.1 and Corollary 1.2 apply.

On the other hand, under the more restrictive condition $\alpha_2^- \leq \alpha \leq \alpha_2^+$, we can make use of inequality (5.7) to get the estimate

$$(5.9) \quad \beta_\gamma \leq \sqrt{\alpha - I(\alpha, 1)} \leq 1.$$

Now recalling the expression of $I(\alpha, 1)$ computed in (4.14) we obtain the explicit estimate, valid for $\gamma \in (-\infty, \gamma_2^-] \cup [\gamma_2^+, +\infty)$:

$$(5.10) \quad \beta_\gamma \leq \sqrt{\frac{3}{\gamma} - \frac{1}{2} - \mu \cdot \frac{\cosh \frac{1}{2} \cosh \mu - 1}{\sinh \frac{1}{2} \sinh \mu}},$$

where μ is the complex number

$$\mu = \frac{1}{2} \sqrt{1 + 4(3 - \gamma)/\gamma}.$$

The choice between the two complex square roots of $1 + 4(3 - \gamma)/\gamma$ does not affect the result and the radical in (5.10) (or in (5.13) below) is nonnegative because of the last inequality in (3.12). Observe that taking here $\gamma = 1$ gives the blowup criterion (1.5) for the Camassa–Holm equation.

If otherwise $\alpha \in [\alpha_1^-, \alpha_1^+] \setminus [\alpha_2^-, \alpha_2^+]$ then, from the inequality (5.3) $I(\alpha, \beta) \geq R(\alpha, \beta)$, we obtain the bound $\beta_\gamma \leq \tilde{\beta}_\gamma$, where $\tilde{\beta}_\gamma$ is the only zero of the quadratic polynomial $\beta \mapsto \beta^2 + R(\alpha, \beta) - \alpha$ inside the interval $[1, \frac{e+1}{e-1}]$. Thus, letting

$$(5.11) \quad \begin{aligned} b &\equiv \frac{e-1}{2} \left(I(\alpha, \frac{e+1}{e-1}) - I(\alpha, 1) \right) \\ &= -\frac{e-1}{2} \left(\frac{(e+1)^2}{2e} \frac{P'_{-1/2+\mu}}{P_{-1/2+\mu}}(\cosh 1) + \frac{1}{2} - \mu \cdot \frac{\cosh \frac{1}{2} \cosh \mu - 1}{\sinh \frac{1}{2} \sinh \mu} \right). \end{aligned}$$

and

$$(5.12) \quad \begin{aligned} c &\equiv \frac{e+1}{2}I(\alpha, 1) - \frac{e-1}{2}I(\alpha, \frac{e+1}{e-1}) - \alpha \\ &= \frac{e+1}{2} \left(-\frac{1}{2} + \mu \cdot \frac{\cosh \frac{1}{2} \cosh \mu - 1}{\sinh \frac{1}{2} \sinh \mu} \right) - \frac{e-1}{2} \left(\frac{(e+1)^2}{2e} \cdot \frac{P'_{-1/2+\mu}(\cosh 1)}{P_{-1/2+\mu}} \right), \end{aligned}$$

we get the estimate, valid for $\gamma \in [\gamma_2^-, \gamma_1^-] \cup \gamma \in [\gamma_1^+, \gamma_2^+]$:

$$(5.13) \quad \beta_\gamma \leq -\frac{b}{2} + \frac{1}{2}\sqrt{b^2 - 4c}.$$

By our construction, for $\gamma = \gamma_2^-$ or $\gamma = \gamma_2^+$, the equality holds in (5.10) and in (5.13) and we have in this case $\beta_{\gamma_2^-} = \beta_{\gamma_2^+} = 1$.

The limit $\beta_\infty = \lim_{\gamma \rightarrow \infty} \beta_\gamma$ is obtained taking $\alpha = -1$ in (5.1). Going back to the estimate (5.10), letting $\gamma \rightarrow \infty$ we get

$$(5.14) \quad \beta_\infty \leq \sqrt{\frac{\sqrt{3}(1 - \cosh \frac{1}{2} \cos \frac{\sqrt{3}}{2})}{2 \sinh \frac{1}{2} \sin \frac{\sqrt{3}}{2}}} - \frac{1}{2}.$$

The three claims of Theorem 2.2 are now established. \square

6. THE CASE OF THE CAMASSA–HOLM EQUATION

In the case of the Camassa–Holm equation ($\alpha = 2$), it is remarkable that the Euler–Lagrange equation

$$(6.1) \quad (\omega_x v_x)_x - 2\omega v = 2\omega, \quad x \in (0, 1),$$

associated with the minimization of $I(2, \beta)$ can be explicitly solved *for any* β .

Indeed, observing that from (4.10) we have $\omega = \omega_{xx}$, on $(0, 1)$, the associate homogeneous equation $(\omega v_x)_x - 2\omega v = 0$ clearly possess $v(x) = \omega_x(x)$ as a solution. We can compute a second independent solution of this homogeneous equation of the form $v(x) = \omega_x f(x)$. Then f must satisfy $f_{xx} + (\frac{2\omega}{\omega_x} + \frac{\omega_x}{\omega})f_x = 0$, provided $\omega\omega_x \neq 0$. Thus, $f_x(x) = \frac{1}{\omega\omega_x^2}$ in intervals where $\omega_x \neq 0$. Observe that

$$\text{if } \beta > \frac{e-1}{e+1}, \text{ then } \omega_x > 0 \text{ on the interval } (0, 1)$$

$$\text{if } 0 < \beta < \frac{e-1}{e+1}, \text{ then } \omega_x \neq 0 \text{ for } x \in (0, 1) \text{ and } x \neq x_\beta \equiv \frac{1}{2} \log\left(\frac{(1-\beta)e}{1+\beta}\right).$$

On the other hand, using again $\omega = \omega_{xx}$ and integration by parts in the indefinite integrals below we see that, for $x \neq x_\beta$ and an arbitrary constant $C \in \mathbb{R}$

$$\begin{aligned} \omega_x(x) \int \frac{1}{\omega\omega_x^2} &= -\omega_x(x) \int \frac{1}{\omega^2} \cdot \left(\frac{1}{\omega_x}\right)_x \\ &= -\frac{1}{\omega(x)^2} + C\omega_x(x) - 2\omega_x(x) \int \frac{1}{\omega^3} \end{aligned}$$

The expression on the right-hand side is well defined on the whole interval $(0, 1)$, for all $0 \leq \beta < \frac{e-1}{e+1}$. Therefore, the general solution of (6.1) is

$$(6.2) \quad v(x) = -1 + \omega_x(x) \left(\lambda + 2\mu \int \frac{1}{\omega^3} \right) + \frac{\mu}{\omega(x)^2}, \quad x \in (0, 1).$$

We compute $\int \frac{1}{\omega^3}$ making the change of variables $y = e^x$:

$$\int \frac{1}{\omega(x)^3} dx = \frac{(e-1)^3}{(1+\beta)^3} \left[\int \frac{8y^2}{(y^2+B)^3} dy \right]_{y=e^x},$$

with

$$B = \frac{e(1-\beta)}{1+\beta}.$$

The last integral can be easily computed distinguishing the cases $B > 0$, $B = 0$ and $B < 0$. For example, in the case $B > 0$, that corresponds to $0 < \beta < 1$, we have

$$\begin{aligned} \int \frac{1}{\omega(x)^3} dx &= \frac{(e-1)^3}{(1+\beta)^3} \left(\frac{y(y^2-B)}{B(y^2+B)^2} + B^{-3/2} \arctan(t/\sqrt{B}) \right) \\ &= \frac{2 \sinh(1/2)^2}{1-\beta^2} \cdot \frac{\omega_x(x)}{\omega(x)^2} + \frac{8 \sinh(1/2)^3}{(1-\beta^2)^{3/2}} \arctan\left(\sqrt{\frac{1+\beta}{e(1-\beta)}} e^x\right). \end{aligned}$$

The minimizer \bar{v} of $I(\alpha, \beta)$, with $0 \leq \beta < \frac{e+1}{e-1}$ is obtained choosing in the above expression the coefficients $\bar{\lambda}$ and $\bar{\mu}$ solving the linear system

$$\begin{cases} \omega_x(0)\bar{\lambda} + \frac{1}{\omega(0)}\bar{\mu} = 1 \\ \omega_x(1)\bar{\lambda} + \left(2\omega_x(1) \left(\frac{1}{\omega(1)^2} + \int_0^1 \frac{1}{\omega^3} \right) \right) \bar{\mu} = 1. \end{cases}$$

Once \bar{v} is computed as indicated, the minimum is given, for $0 \leq \beta < \frac{e+1}{e-1}$, by

$$\begin{aligned} (6.3) \quad I(2, \beta) &= 2 + J(\bar{v}) = \omega(1)\bar{v}_x(1) - \omega(0)\bar{v}_x(0) \\ &= \omega(1)^2 \left(\bar{\lambda} + 2\bar{\mu} \int_0^1 \frac{1}{\omega^3} \right) - \bar{\lambda}\omega(0). \end{aligned}$$

The explicit expression of $I(2, \beta)$ is thus easily written in terms of elementary functions. We do not reproduce the complete formula here as it is too long to be really useful. We just provide here a few particular values and its plot in the interval $[0, \frac{e+1}{e-1}]$ (see Figure 6).

$$\begin{aligned} (6.4) \quad I(2, 0) &= 1 + \frac{\arctan(\sinh \frac{1}{2})}{\sinh \frac{1}{2} + \arctan(\sinh \frac{1}{2})(\sinh \frac{1}{2})^2} = 1.737\dots \\ I(2, 1) &= -\frac{1}{2} + \frac{3}{2} \cdot \frac{\cosh \frac{1}{2} \cosh \frac{3}{2} - 1}{\sinh \frac{1}{2} \sinh \frac{3}{2}} = 1.734\dots \\ I(2, \frac{e+1}{e-1}) &= \frac{(e+1)^2}{e^2+1} = 1.648\dots \end{aligned}$$

The value of $I(2, 0)$ was computed also in [29] or [33].

Remark 6.1. The last formula in (6.4), in fact, is obtained in a slight different way, as in our previous computations we excluded the limit case $\beta = \frac{e+1}{e-1}$. Of course, this formula agrees with that obtained from the more general one (4.9) for $\alpha = 2$, as $\nu(2) = 1$ and $P_1(y) = y$. But we can easily prove it without relying on the more complicated approach used to get (4.9). Indeed, when $\beta = \frac{e+1}{e-1}$, we can write the general solution of (6.1) as

$$(6.5) \quad v_{\lambda, \mu}(x) = -1 + \lambda \cosh(x) + \mu \left(1 + \frac{1}{2} \cosh(x) \log \left(\frac{\cosh(x) - 1}{\cosh(x) + 1} \right) \right), \quad x \in (0, 1).$$

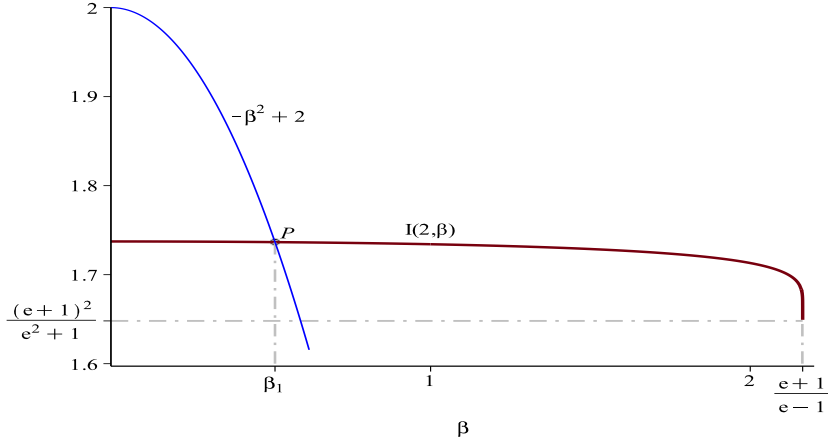


FIGURE 7. The plot of $I(2, \beta)$ and the construction of $\beta_1 = 0.513\dots$

The appropriate boundary condition for $\beta = \frac{e+1}{e-1}$ is $v_{\lambda, \mu} \in E_{(e+1)/(e-1), 0}$. The only possibility to achieve this is to choose $\mu = 0$ (recall that the elements of $E_{(e+1)/(e-1), 0}$ are $O(\sqrt{|\log(x)|})$ as $x \rightarrow 0+$). The remaining condition $v(1) = 0$ requires $\lambda = 1/\cosh(1)$. The minimizer of the functional J in the case $\alpha = 2$ and $\beta = \frac{e+1}{e-1}$, is then

$$(6.6) \quad \bar{v}(x) = -1 + \frac{\cosh(x)}{\cosh(1)}.$$

This indeed leads to the last formula in (6.4).

In view of the application of Theorem 2.1, it is interesting to make the numerical study of the zeros of the function $\beta \mapsto \beta^2 + I(2, \beta) - 2$, on the interval $[0, \frac{e+1}{e-1}]$. The only zero is $\beta_1 = 0.513\dots$. According to Theorem 2.1, solutions of Camassa–Holm blowup provided $u'_0(x_0) < -\beta_1|u_0(x_0)|$. On the other hand, Theorem 2.2 predicted $\beta_1 \leq 0.515\dots$ (see criterion (1.5)). This is another confirmation that the estimate provided by Theorem 2.2 is almost sharp.

7. PROOF OF COROLLARY 1.2 AND THE CASE OF NON-PERIODIC SOLUTIONS.

Proof of Corollary 1.2. We assume that $\gamma \geq \gamma_1^+$, so in particular that $\gamma > 0$. In the case $\gamma \leq \gamma_1^-$ the proof is similar. From the assumption of the corollary, $T^* = +\infty$. Then Theorems 2.1-2.2 apply, implying that for all $t \geq 0$, and all $x \in \mathbb{R}$ we have $u_x(t, x) \geq -\beta_\gamma|u(t, x)|$. If $[a, b]$ is an interval where $u(t, x) \geq 0$ for all $x \in [a, b]$, then we have

$$\begin{aligned} - \int_a^b \beta_\gamma e^{\beta_\gamma x} u(t, x) dx &\leq \int_a^b e^{\beta_\gamma x} u_x(t, x) dx \\ &= e^{\beta_\gamma b} u(t, b) - e^{\beta_\gamma a} u(t, a) - \beta_\gamma \int_a^b e^{\beta_\gamma x} u(t, x) dx. \end{aligned}$$

This implies that $e^{\beta_\gamma a} u(t, a) \leq e^{\beta_\gamma b} u(t, b)$.

If $u(t, \cdot) \leq 0$ on $[a, b]$, then applying again Theorem 2.1 we have $u_x(t, x) \geq \beta_\gamma u(t, x)$. This in turn implies

$$\begin{aligned} \int_a^b \beta_\gamma e^{-\beta_\gamma x} u(t, x) \, dx &\leq \int_a^b e^{-\beta_\gamma x} u_x(t, x) \, dx \\ &= e^{-\beta_\gamma b} u(t, b) - e^{-\beta_\gamma a} u(t, a) + \beta_\gamma \int_a^b e^{-\beta_\gamma x} u(t, x) \, dx. \end{aligned}$$

We then conclude that $e^{-\beta_\gamma a} u(t, a) \leq e^{-\beta_\gamma b} u(t, b)$.

Summarizing, we proved that $x \mapsto e^{\beta_\gamma x} u(x, t)$ is monotone increasing in any interval where $u(\cdot, t) \geq 0$ and $x \mapsto e^{-\beta_\gamma x} u(x, t)$ is monotone increasing in any interval where $u(\cdot, t) \leq 0$. The condition $u(x_0, t_0) = 0$ together with the periodicity of u imply that $u(\cdot, t_0) \equiv 0$ and so $u \equiv 0$ because of the conservation of the H^1 -norm. \square

A digression on non-periodic solutions. The above proof can also be applied to non-periodic solutions of the rod equations, defined on the whole real line and with vanishing boundary conditions at $\pm\infty$, but with $x \in \mathbb{R}$ and in this case the kernel p of $(1 - \partial_x^2)^{-1}$ is given by

$$p(x) = \frac{1}{2} e^{-|x|}.$$

The equation is written in the same way as in (1.1). Let us recall that by the result in [3], if $\gamma \in [1, 4]$ and $u_0 \in H^s(\mathbb{R})$, with $s > 3/2$, is such that

$$(7.1) \quad \exists x_0 \in \mathbb{R} \quad \text{such that} \quad u'_0(x_0) < -\sqrt{-\frac{1}{2} + \frac{3}{\gamma} - \frac{\sqrt{12 - 3\gamma}}{2\sqrt{\gamma}}} |u(x_0)|$$

then the unique local-in-time solution $u \in C([0, T], H^s(\mathbb{R})) \cap C^1([0, T], H^{s-1}(\mathbb{R}))$ must blow up in finite time. In the Camassa–Holm case $\gamma = 1$ the above coefficient equals -1 and is known to be optimal, see [19]. We then obtain the following version of Corollary 1.2 in the case of non-periodic solution (we only have to reproduce the above proof taking

$$(7.2) \quad \beta_\gamma = \sqrt{-\frac{1}{2} + \frac{3}{\gamma} - \frac{\sqrt{12 - 3\gamma}}{2\sqrt{\gamma}}}$$

equal to the opposite of the coefficient on the right-hand side of (7.1)):

Corollary 7.1. *Let $\gamma \in [1, 4]$ and $s > 3/2$. The identically zero solution is the only global solution of the rod equation (1.1) (with $p(x) = \frac{1}{2} e^{-|x|}$ and $x \in \mathbb{R}$) in $C([0, +\infty), H^s(\mathbb{R})) \cap C^1([0, +\infty), H^{s-1}(\mathbb{R}))$, satisfying at least one of the following two conditions at some time $t_0 \geq 0$:*

- i) $u(t_0, \cdot)$ vanishes at two distinct points $x_1, x_2 \in \mathbb{R}$ or
- ii) $\liminf_{x \rightarrow +\infty} e^{\beta_\gamma x} u(t_0, x) \leq 0$ and $\limsup_{x \rightarrow -\infty} e^{-\beta_\gamma x} u_0(t_0, x) \geq 0$, where β_γ is given by equation (7.2).

In particular, if $0 \neq u_0 \in H^s(\mathbb{R})$ is such that $u_0(x) = o(e^{\beta_\gamma |x|})$ for $|x| \rightarrow \infty$, then the corresponding solution of the rod equation must blow up in finite time.

Remark 7.2. In the last conclusion of the corollary we recover the known fact that solutions of the Camassa–Holm equation arising from compactly supported data (see [20]), or more in general (see [2]) from data decaying faster than the peakons —solitons with profile $ce^{-|x|}$ — feature a wave breaking phenomenon after some time. The proofs given in [2, 20] relied on

McKean's necessary and sufficient condition for wave breaking and it was not suitable for the generalization to $\gamma \neq 1$.

The method that we used in our previous paper [3] in the non-periodic case looks simpler than that of the present paper. Indeed, we relied on elementary estimates for bounding from below the convolution term $p * (\alpha u^2 + u_x^2)$ without making use of variational methods. On the other hand, the result obtained therein is much weaker as the range of applicability for the parameter γ is considerably narrower. We point out that applying the variational method of the present paper in the non-periodic case would not give better results than in [3]. Indeed, the main issue would be the study of the minimization problem (the analogue of (2.1) but now with $p(x) = \frac{1}{2}e^{-|x|}$):

$$(7.3) \quad I_{\mathbb{R}}(\alpha, \beta) = \inf \left\{ \int_{\mathbb{R}} (p + \beta p_x) (\alpha u^2 + u_x^2) dx : u \in H^1(\mathbb{R}), u(0) = 1 \right\}.$$

The analogue of Proposition (3.3) and Proposition 3.4 in our present setting is provided by the following one:

Proposition 7.3. *We have*

$$(7.4) \quad I_{\mathbb{R}}(\alpha, \beta) > -\infty \iff \alpha \geq -\frac{1}{4} \quad \text{and} \quad -1 \leq \beta \leq 1.$$

Moreover, the function $\beta \mapsto I_{\mathbb{R}}(\alpha, \beta)$ is constant on the interval $[-1, 1]$ and under the above restrictions on α and β we have

$$I_{\mathbb{R}}(\alpha, \beta) = -\frac{1}{2} + \frac{1}{2}\sqrt{1 + 4\alpha}.$$

The proof of this proposition, that we only sketch, relies on the identity $p(x) + \beta p'(x) = \frac{1+\beta}{2}e^{-x}\mathbf{1}_{\mathbb{R}^-} + \frac{1-\beta}{2}e^{-x}\mathbf{1}_{\mathbb{R}^+}$, implying that

$$I_{\mathbb{R}}(\alpha, \beta) = \inf \left\{ \int_0^{\infty} e^{-x} (\alpha u^2 + u_x^2) dx : u \in H^1(\mathbb{R}^+), u(0) = 1 \right\}.$$

For $\alpha > 0$, one easily find solving the associate Euler–Lagrange equation the minimizer $\bar{u} \in H^1(\mathbb{R}^+)$ given by $\bar{u}(x) = \exp(\frac{1}{2} - \frac{1}{2}\sqrt{1 + 4\alpha})$.

For $-\frac{1}{4} \leq \alpha \leq 0$, \bar{u} does not belong to $H^1(\mathbb{R}^+)$ but one can construct a minimizing sequence taking $u_{\ell} = \bar{u}$ on $[0, \ell]$ and $u_{\ell}(x) = C_{\ell} \exp(\sqrt{-\alpha}x)$ for $x \geq \ell$, with C_{ℓ} chosen in a such way that u_{ℓ} is continuous. In this way, $\int_{\ell}^{+\infty} e^{-x} (\alpha u^2 + u_x^2) dx = 0$ and letting $\ell \rightarrow \infty$ we find $\int_0^{\ell} e^{-x} (\alpha u^2 + u_x^2) dx = -\frac{1}{2} + \frac{1}{2}\sqrt{1 + 4\alpha}$.

Notice that the the condition $\alpha \geq -1/4$ gives the restriction $\gamma \leq 4$. On the other hand, the formula $\beta_{\gamma} = \inf\{\beta \geq 0 : \beta^2 + I_{\mathbb{R}}(\alpha, \beta) - \alpha \geq 0\}$ immediately gives $\beta_{\gamma} = +\infty$ if $\gamma < 1$. We recover in this way that we must restrict ourselves to $\gamma \in [1, 4]$ and that β_{γ} is given by formula (7.2) in the non-periodic case.

8. APPENDIX: NUMERICAL ANALYSIS

In this appendix we briefly revisit Theorem 2.1 using a numerical approach. Rather than obtaining estimates for β_{γ} by making use of some exact formulas, as we did in Theorem 2.2, we can evaluate β_{γ} numerically. In order to achieve this, we need first to approximate $I(\alpha, \beta)$. This can be done by approaching the solutions of the boundary value problem (4.11), for any fixed α

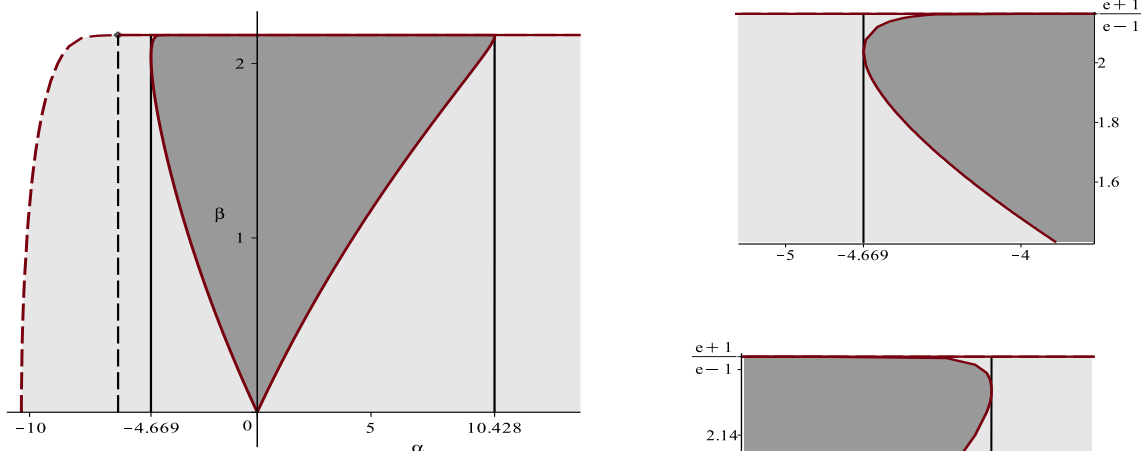


FIGURE 8. Set of points (α, β) such that $\beta \geq 0$ and $\beta^2 + I(\alpha, \beta) - \alpha \geq 0$ (dark gray region). Right: two zooms on the upper-left and upper-right corners.

and β , with $0 \leq \beta < \frac{e+1}{e-1}$. This can be done with arbitrary high precision applying *e.g.* a finite difference technique with Richardson extrapolation. In order to get an error smaller than 0.001 we needed a large number of grid points (a few thousands), especially when β is getting close to the critical value $\frac{e+1}{e-1}$.

The following table provides the values of β_γ computed numerically, corresponding to the values of γ listed in [18] and associated with known hyper-elastic materials. Only in one case ($\gamma = -0.539$) our theorems are not applicable.

γ	-29.476	-4.891	-2.571	-1.646	-0.539	1.010	1.236	1.700	2.668	3.417
β_γ	0.326	0.492	0.684	0.933	n.a.	0.507	0.375	.0207	0.035	0.035

Picture 8 illustrates the set of points (α, β) such that $\beta \geq 0$ and $\beta^2 + I(\alpha, \beta) - \alpha \geq 0$. It turns out that this set is nonempty (approximatively) for $\alpha \in [-4.669, 10.428]$. Recalling the definition of β_γ given in (5.1)-(5.2), we see that Theorem 2.1 can be applied for $\gamma \in (-\infty, -0.817\dots] \cup [0.262\dots, +\infty)$, which is a range slightly larger than the range $\gamma \in (-\infty, \gamma_2^-] \cup [\gamma_2^+, \infty)$, predicted by Theorem 2.2. Relying in this numerical analysis, we claim that Corollary 1.2 remains true in such a larger range.

Figure 8 provides the plot of the the function $\gamma \mapsto \beta_\gamma$ as computed numerically. The curve is very close to that in Figure 2 and the two curves almost overlap, in particular for $\gamma > \gamma_1^+$ or $\gamma < \gamma_2^+$ (only when $\gamma \in [\gamma_2^-, \gamma_1^-]$ the difference between the two curves becomes visible).

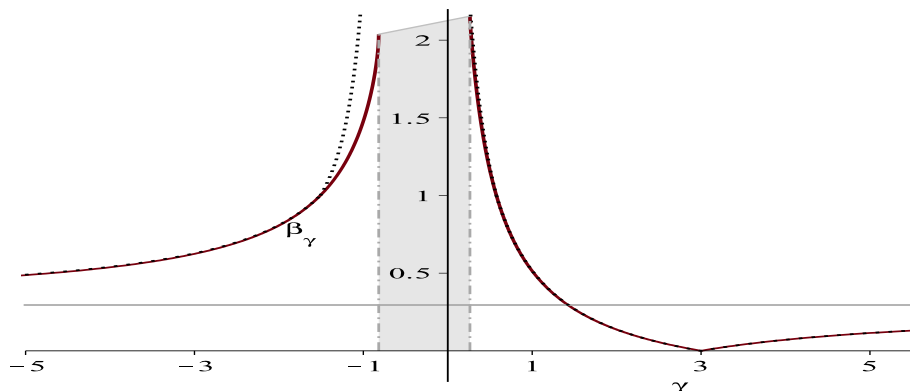


FIGURE 9. The plot of β_γ in the interval $\gamma \in (-\infty, -0.817] \cup [0.262, +\infty)$ (continuous line) and the upper estimate of β_γ provided by Theorem 2.2 (dotted line).

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